Three Essays on Advances in Macroeconomic Theory and Modeling

By

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Abstract

My dissertation consists of three papers on bifurcation and market game models. My research focuses on understanding bifurcation phenomena of macroeconometric models, exploring price stickiness and markup variations in market game models with production through strategic interaction, and analyzing the possibility of endogenous business cycles in the overlapping generation market game models with production. Specifically, the three chapters are:

Bifurcation of Macroeconometric Models and Robustness of Dynamical Inferences is a survey paper I coauthored with Prof. Barnett (Barnett and Chen (2015)). In systems theory, it is well known that the parameter spaces of dynamical systems are stratified into bifurcation regions, with each supporting a different dynamical solution regime. Some can be stable, with different characteristics, such as monotonic stability, periodic damped stability, or multiperiodic damped stability, and some can be unstable, with different characteristics, such as periodic, multiperiodic, or chaotic unstable dynamics. But in general the existence of bifurcation boundaries is normal and should be expected from most dynamical systems, whether linear or nonlinear. Bifurcation boundaries in parameter space are not evidence of model defect. While existence of such bifurcation boundaries is well known in economic theory, econometricians using macroeconometric models rarely take bifurcation into consideration, when producing policy simulations from macroeconometric models. Such models are routinely simulated only at the point estimates of the models' parameters.

Barnett and He (1999) explored bifurcation stratification of Bergstrom and Wymer's (1976) continuous time UK macroeconometric model. Bifurcation boundaries intersected the confidence region of the model's parameter estimates. Since then, Barnett and his coauthors have been conducting similar studies of many other newer macroeconometric models

spanning all basic categories of those models. So far, they have not found a single case in which the model's parameter space was not subject to bifurcation stratification. In most cases, the confidence region of the parameter estimates were intersected by some of those bifurcation boundaries. The most fundamental implication of this research is that policy simulations with macroeconometric models should be conducted at multiple settings of the parameters within the confidence region. While this result would be as expected by systems theorists, the result contradicts the normal procedure in macroeconometrics of conducting policy simulations solely at the point estimates of the parameters.

This survey provides an overview of the classes of macroeconometric models for which these experiments have so far been run and emphasizes the implications for lack of robustness of conventional dynamical inferences from macroeconometric policy simulations. By making this detailed survey of past bifurcation experiments available, we hope to encourage and facilitate further research on this problem with other models and to emphasize the need for simulations at various points within the confidence regions of macroeconometric models, rather than at only point estimates.

Price Stickiness and Markup Variations in Market Games is a paper I coauthored with Prof. Stephen Spear and Dr. C. Gizem Korpeoglu (Chen et al. (2017)). Shapley-Shubik market game model received quite a bit of attention in the general equilibrium literature of the 1980's and 1990's, but never caught on as a possible alternative to models of monopolistic competition in macroeconomics. In this paper, we suggest that the market game model can provide a better micro-foundation for new Keynesian general equilibrium analysis than existing models based on monopolistic competition. We show that the market game generates equilibria that have two important features. First, we show that when firms have market power, their market-shares in both input and output markets affect the first-order conditions of their best responses, in ways that resemble the effects of price changes. From this observation, we are able to establish that firm quantity adjustments (holding input prices fixed) can maintain the Nash equilibrium of the model in versions of the model that exhibit indeterminacy of the Nash equilibrium. Hence, these versions of the model naturally admit sticky prices, regardless of the mechanism(s) that might lead firms to want to keep input prices unchanging. Second, we show that there is a close relationship between any individual firm's markup of price over marginal cost and its market share. What the market game brings to the discussion of markups that is new, is the fact that markets populated by finite numbers of firms operating under possibly different technologies will generate data on markup movements over different equilibria that can vary positively, negatively, variably, or not at all over business-cycle-like expansions and contractions.

Endogenous Business Cycles in the Overlapping Generations Market Game Model is my job market paper (Chen (2018)). We then extend the analysis on market game models with production in chapter two to an overlapping generations (OLG) market game model, and study whether strategic interactions contribute to instabilities of the economic dynamics. Grandmont (1985) was one of the first papers to raise the possibility that endogenous complex dynamics might provide an alternative explanation for business cycle fluctuations, by showing that such dynamics could arise in conventional OLG models, although only for the case of sufficiently large risk aversion on the part of old agents in the model. Goenka et al. (1998) showed in the context of a pure exchange OLG market game model that the nonlinearities introduced by imperfect competition were such that one could obtain chaotic dynamics even for log utility, as long as markets were thin in terms of amount of endowment agents offered. Goenka et al. (1998) note that extensions of their work with this kind of model suggests that production smooths the model in the sense that complex dynamics are not as easily generated as in the pure exchange model. In this paper, analysis shows that production combined with price-taking behavior by households locks down the ratio of output and input prices, which then reduces the nonlinearity that arises in the pure exchange model. Specifically, we show in the paper that when incorporating production in the market game OLG model, the price dynamics depend on market thickness, general equilibrium price ratios, individual offers and particular choices of utility function. We find that for complex dynamics to occur, the preferences in our model must be a mix of preferences, for example, a combination of preferences with constant relative risk aversions and increasing relative risk aversions. We also show the impossibility of such price dynamics to occur for log-linear preferences. In other words, the case for complex dynamics to occur with particular production functions and utility functions is much more limited. As a result, complex dynamics are not as easily observable as in models without production. Finally, we are able to confirm the results from Goenka et al. (1998) on the Pareto rankability of Nash equilibria in terms of market thickness, which has important welfare implications for business cycle-like activity based on the coordination equilibria that can arise in market game models.

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Chapter 1: Bifurcation of Macroeconometric Models and Robustness of Dynamical Inferences[1]

1.1 Bifurcation Of Macroeconomic Models[2]

1.1.1 Introduction

Bifurcation has long been a topic of interest in dynamical macroeconomic systems. Bifurcation analysis is important in understanding dynamic properties of macroeconomic models as well as in selection of stabilization policies. The goal of this survey is to summarize work by William A. Barnett and his coauthors on bifurcation analyses in macroeconomic models to facility and motivate work by others on further models. In section 1.1, we introduce the concept of bifurcation and its role in studies of macroeconomic systems and also discuss several types of bifurcations by providing examples summarized from Barnett and He (2004, 2006b). In sections 1.2-1.8, we discuss bifurcation analysis and approaches with models from Barnett's other papers on this subject.

To explain what bifurcation is, Barnett and He (2004,2006b) begin with the general form of many existing macroeconomic models:

$$\mathbf{Dx} = \mathbf{f}(\mathbf{x}, \boldsymbol{\theta}), \tag{1.1.1}$$

where $\mathbf{D}$ is the vector-valued differentiation operator, $\mathbf{x}$ is the state vector, $\boldsymbol{\theta}$ is the parameter vector, and $\mathbf{f}$ is the vector of functions governing the dynamics of the system, with each component assumed to be smooth in a local region of interest.

[1] This paper is published as Barnett and Chen (2015).

[2] This section is summarized from Barnett and He (2004,2006b).

In system (1.1.1), the focus of interest lies in the settings of the parameter vector, $\boldsymbol{\theta}$. Assume $\boldsymbol{\theta}$ takes values within a theoretically feasible set Θ. The value of $\boldsymbol{\theta}$ can affect the dynamics of the system substantially through a small change, and we say a bifurcation occurs in the system, if such a small change in parameters fundamentally alters the nature of the dynamics of the system. In particular, bifurcation refers to a change in qualitative features instead of quantitative features of the solution dynamics. A change in quantitative features of dynamical solutions may refer to a change in such properties as the period or amplitude of cycles, while a change in qualitative features may refer to such changes as changes from one type of stability or instability to another type of stability or instability.

A point within the parameter space at which a change in qualitative features of the dynamical solution path occurs defines a point on a bifurcation boundary. At the bifurcation point, the structure of the dynamic system may change fundamentally. Different dynamical solution properties can occur when parameters are close to but on different sides of a bifurcation boundary. A parameter set can be stratified by bifurcation boundaries into several subsets with different types of dynamics within each subset.

There are several types of bifurcation boundaries, such as Hopf, pitchfork, saddle-node, transcritical, and singularity bifurcation. Each type of bifurcation produces a different type of qualitative dynamic change. We illustrate these different types of bifurcation by providing examples in section 1.1.3. Bifurcation boundaries have been discovered in many macroeconomic systems. For example, Hopf bifurcations have been found in growth models (e.g., Benhabib and Nishimura (1979), Boldrin and Woodford (1990), Dockner and Feichtinger (1991), and Nishimura and Takahashi (1992)) and in overlapping generations models. Pitchfork bifurcations have been found in the tatonnement process (e.g., Bala (1997) and Scarf (1960)). Transcritical bifurcations have been found in Bergstrom and Wymer's (1976) UK model (Barnett and He (1999)) and singularity bifurcation in Leeper and Sims' Euler-equation model (Barnett and He (2008)).

One reason we are concerned about bifurcation phenomena in macroeconomic models is because changes in parameters could affect dynamic behaviors of the models and consequently the outcomes of imposition of policy rules. For example, Bergstrom and Wymer's (1976) UK model operates close to bifurcation boundaries between stable and unstable regions of the parameter space. In this case, if a bifurcation boundary intersects the confidence region of the parameter estimates, different qualitative properties of solution can exist within this confidence region. As a result, robustness of inferences about dynamics can be damaged, especially if inferences about dynamics are based on model simulations with the parameters set only at their point estimates. When confidence regions are stratified by bifurcation boundaries, dynamical inferences need to be based on simulations at points within each of the stratified subsets of the confidence region.

Knowledge of bifurcation boundaries is directly useful in policy selection. If the system is unstable, a successful policy would bifurcate the system from the unstable to stable region. In that sense, stabilization policy can be viewed as bifurcation selection. As illustrated in section 1.1.2, Barnett and He (2002) have shown that successful bifurcation policy selection can be difficult to design.

Barnett's work has found bifurcation phenomena in every macroeconomic model that he and his coauthors have so far explored. Barnett and He (1999,2002) examined the dynamics of Bergstrom-Wymer's continuous-time dynamic macroeconomic model of the UK economy and found both transcritical and Hopf bifurcation boundaries. Barnett and He

(2008) estimated and displayed singularity bifurcation boundaries for the Leeper and Sims (1994) Euler equations model. Barnett and Duzhak (2010) found Hopf and period doubling bifurcations in a New Keynesian model. Banerjee, Barnett, Duzhak and Gopalan (2011) examined the possibility of cyclical behavior in the Marshallian Macroeconomic Model. Barnett and Eryilmaz (2013, 2014) investigated bifurcation in open economy models. Barnett and Ghosh (2013a) investigated the existence of bifurcations in endogenous growth models.

This survey is organized in the chronological order of Barnett's work on bifurcation of macroeconomic models, from early models to many of the most recent models.

1.1.2 Stability

There are two possible approaches to analyze bifurcation phenomena: global and local. Methods in Barnett's current papers have used local analysis, which is analysis of the linearized dynamic system in a neighborhood of the steady state. In his papers, equation (1.1.1) is linearized in the form

$$\mathbf{Dx} = \mathbf{A}(\boldsymbol{\theta})\mathbf{x} + \mathbf{F}(\mathbf{x}, \boldsymbol{\theta}), \tag{1.1.2}$$

where $\mathbf{A}(\boldsymbol{\theta})$ is the Jacobian matrix of $\mathbf{f}(\mathbf{x}, \boldsymbol{\theta})$, and $\mathbf{F}(\mathbf{x}, \boldsymbol{\theta}) = \mathbf{f}(\mathbf{x}, \boldsymbol{\theta}) - \mathbf{A}(\boldsymbol{\theta})\mathbf{x} = \mathrm{o}(\mathbf{x}, \boldsymbol{\theta})$ is the vector of higher order term. Define $\mathbf{x}^*$ to be the system's steady state equilibrium, such that $\mathbf{f}(\mathbf{x}^*, \boldsymbol{\theta}) = \mathbf{0}$, and redefine the variables such that the steady state is the point $\mathbf{x}^* = \mathbf{0}$ by replacing $\mathbf{x}$ with $\mathbf{x} - \mathbf{x}^*$.

The local stability of (1.1.1), for small perturbation away from the equilibrium, can be studied through the eigenvalues of $\mathbf{A}(\boldsymbol{\theta})$, which is a matrix-valued function of the parameters $\boldsymbol{\theta}$. It is important to know at what parameter values, $\boldsymbol{\theta}$, the system (1.1.1) is unstable. But it is also important to know the nature of the instability, such as periodic, multiperiodic, or chaotic, and the nature of the stability, such as monotonically convergent, damped single-periodic convergent, or damped multiperiodic convergent. For global analysis, which can be far more complicated than local analysis, higher order terms must be considered, since the perturbations away from the equilibrium can be large. Analysis of $\mathbf{A}(\boldsymbol{\theta})$ alone may not be adequate. More research on global analysis of macroeconomic models is needed.

To analyze the local stability properties of the system, we need to locate the bifurcation boundaries. The boundaries must satisfy

$$det\big(\mathbf{A}(\boldsymbol{\theta})\big) = 0. \tag{1.1.3}$$

According to Barnett and He (2004), if all eigenvalues of $\mathbf{A}(\boldsymbol{\theta})$ have strictly negative real parts, then (1.1.1) is locally asymptotically stable in the neighborhood of $\mathbf{x} = \mathbf{0}$. If at least one of the eigenvalues of $\mathbf{A}(\boldsymbol{\theta})$ has positive real part, then (1.1.1) is locally asymptotically unstable in the neighborhood of $\mathbf{x} = \mathbf{0}$.

The bifurcation boundaries can be difficult to locate. In Barnett and He (1999, 2002), various methods are applied to locate the bifurcation boundaries characterized by (1.1.3). Equation (1.1.3) usually cannot be solved in closed form, when $\boldsymbol{\theta}$ is multi-dimensional. As a result, numerical methods are extensively used for solving (1.1.3).

Before proceeding to the next section, we introduce the definition of hyperbolic for flows and maps, respectively. According to Hale and Kocak (1991), the following definitions apply.

Definition 1.1.1. An equilibrium point $\mathbf{x}^*$ of $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ is said to be *hyperbolic*, if all the eigenvalues of the Jacobian matrix $D\mathbf{f}(\mathbf{x}^*)$ have nonzero real parts.

Definition 1.1.2 A fixed point $\mathbf{x}^*$of $\mathbf{x} \longmapsto \mathbf{f}(\mathbf{x})$ is said to be *hyperbolic*, if the linear C^1 map $\mathbf{x} \longmapsto D\mathbf{f}(\mathbf{x}^*)\mathbf{x}$ is *hyperbolic*; that is, if the Jacobian matrix $D\mathbf{f}(\mathbf{x}^*)$ at $\mathbf{x}^*$ has no eigenvalues with modulus one.

Definition 1.1.2. refers to discrete-time dynamical systems. Since bifurcations can only occur in a local neighborhood of non-hyperbolic equilibria, we are more interested in the behavior at non-hyperbolic equilibria.

For a discrete-time dynamical system, consider a generic smooth one-parameter family of maps $\mathbf{x} \longmapsto \mathbf{f}(\mathbf{x}, \alpha) = \mathbf{f}_{(\alpha)}(\mathbf{x}),\ \mathbf{x} \in R^n, \alpha \in R.$ Since local bifurcation happens only at nonhyperbolic fixed points, there are three critical cases to consider:

(a) The fixed point $\mathbf{x}^*$ has eigenvalue 1.

(b) The fixed point $\mathbf{x}^*$ has eigenvalue -1.

(c) The fixed point $\mathbf{x}^*$ has a pair of complex-conjugate eigenvalues $e^{\pm i\theta_0}$ with $0 < \theta_0 < \pi.$

The codimension 1 bifurcation associated with case (a) is called a *fold* (*saddle node*) bifurcation. The codimension 1 bifurcation associated with case (b) is called a *flip* (*period doubling*) bifurcation, while the codimension 1 bifurcation associated with case (c) is called a *Neimark-Sacker* bifurcation. *Neimark-Sacker* bifurcation is the equivalent of *Hopf* bifurcation for maps.

In the following section, we are going to introduce three important one-dimensional equilibrium bifurcations described locally by ordinary differential equations. They are transcritical, pitchfork, and saddle-node bifurcations.

1.1.3 Types of Bifurcations

1.1.3.1 Transcritical Bifurcations

For a one-dimensional system,

$$Dx = G(x, \theta),$$

the transversality conditions for a transcritical bifurcation at $(x, \theta) = (0,0)$ are

$$G(0,0) = G_x(0,0) = G_\theta(0,0) = 0,\ G_{xx}(0,0) \neq 0, \text{ and } {G_{\theta x}}^2 - G_{xx}G_{\theta\theta}(0,0) > 0. \quad (1.1.4)$$

An example of such a form is

$$Dx = \theta x - x^2. \quad (1.1.5)$$

The steady state equilibria of the system are at $x^* = 0$ and $x^* = \theta$. It follows that system (1.1.5) is stable around the equilibrium $x^* = 0$ for $\theta < 0$, and unstable for $\theta > 0$. System (1.1.5) is stable around the equilibrium $x^* = \theta$ for $\theta > 0$, and unstable for $\theta < 0$. The nature of the dynamics changes as the system bifurcates at the origin. This transcritical bifurcation arises in systems in which there is a simple solution branch, corresponding here to $x^* = 0$.

Transcritical bifurcations have been found in high-dimensional continuous-time macroeconomic systems, but in high dimensional cases, transversality conditions have to be verified on a manifold. Details are provided in Guckenheimer and Holmes (1983).

1.1.3.2 Pitchfork Bifurcations

For a one-dimensional system,

$$Dx = f(x, \theta).$$

Suppose that there exists an equilibrium x^* and a parameter value θ^* such that (x^*, θ^*) satisfies the following conditions:

$$(a) \quad \frac{\partial f(x, \theta^*)}{\partial x}|_{x=x^*} = 0,$$

$$(b) \quad \frac{\partial^3 f(x, \theta^*)}{\partial x^3}|_{x=x^*} \neq 0,$$

$$(c) \quad \frac{\partial^2 f(x, \theta)}{\partial x \partial \theta}|_{x=x^*, \theta=\theta^*} \neq 0,$$

then (x^*, θ^*) is a pitchfork bifurcation point.

An example of such form is

$$Dx = \theta x - x^3.$$

The steady state equilibria of the system are at $x^* = 0$ and $x^* = \pm\sqrt{\theta}$. It follows that the system is stable when $\theta < 0$ at the equilibrium $x^* = 0$, and unstable at this point when $\theta > 0$. The two other equilibria $x^* = \pm\sqrt{\theta}$ are stable for $\theta > 0$. The equilibrium $x^* = 0$ loses stability, and two new stable equilibria appear. This pitchfork bifurcation, in which a stable solution branch bifurcates into two new equilibria as θ increases, is called a supercritical bifurcation.

Bala (1997) shows how pitchfork bifurcation can occur in the tatonnement process.

1.1.3.3 Saddle-Node Bifurcations

For a one-dimensional system,

$$Dx = f(x, \theta).$$

A saddle-node point (x^*, θ^*) satisfies the equilibrium condition $f(x^*, \theta^*) = 0$ and the Jacobian condition $\frac{\partial f(x, \theta^*)}{\partial x}\Big|_{x=x^*} = 0$, as well as the transversality conditions, as follows:

$$(a) \quad \frac{\partial f(x, \theta)}{\partial \theta}\Big|_{x=x^*, \theta=\theta^*} \neq 0,$$

$$(b) \quad \frac{\partial^2 f(x, \theta)}{\partial x^2}\Big|_{x=x^*, \theta=\theta^*} \neq 0.$$

Sotomayor (1973) shows that transversality conditions for high-dimensional systems can also be formulated.

A simple system with a saddle-node bifurcation is

$$Dx = \theta - x^2.$$

The equilibria are at $x^* = \pm\sqrt{\theta}$, which requires θ to be nonnegative. Therefore, there exist no equilibria for $\theta < 0$, and there exist two equilibria at $x^* = \pm\sqrt{\theta}$, when $\theta > 0$. It follows that when $\theta > 0$, the system is stable at $x^* = \sqrt{\theta}$ and unstable at $x^* = -\sqrt{\theta}$. In this example, bifurcation occurs at the origin as θ increases through zero, which is called the (supercritical) saddle node.

1.1.3.4 Hopf Bifurcations

Hopf bifurcation is the most studied type of bifurcation in economics. For continuous time systems, Hopf bifurcation occurs at the equilibrium points at which the system has a Jacobian matrix with a pair of purely imaginary eigenvalues and no other eigenvalues which have zero real parts. For discrete time system, the following theorem applies in the special case of n=2. The Hopf Bifurcation Theorem in Gandolfo (2010, ch. 24, p.497) is widely applied to find the existence of Hopf bifurcation.

Theorem 1.1.1 (*Existence of Hopf Bifurcation in 2 dimensions*) Consider the two-dimensional non-linear difference system with one parameter

$$\mathbf{y}_{t+1} = \boldsymbol{\varphi}(\mathbf{y}_t, \alpha),$$

and suppose that for each α in the relevant interval there exists a smooth family of equilibrium points, $\mathbf{y}_e = \mathbf{y}_e(\alpha)$, at which the eigenvalues are complex conjugates, $\lambda_{1,2} = \theta(\alpha) + i\omega(\alpha)$. If there is a critical value α_0 of the parameter such that

a. the eigenvalues' modulus becomes unity at α_0, but the eigenvalues are not roots of unity (from the first up to the fourth), namely

   $|\lambda_{1,2}(\alpha_0)| = \sqrt{\theta^2 + \omega^2} = 1,\ \ \lambda_{1,2}^j(\alpha_0) \neq 1$ for $j = 1,2,3,4,$

and

b. $\dfrac{d|\lambda_{1,2}(\alpha)|}{d\alpha}\Big|_{\alpha=\alpha_0} \neq 0,$ then there is an invariant closed curve bifurcating from α_0.

This theorem only applies with a 2×2 Jacobian. The earliest theoretical works on Hopf bifurcation include Poincaré (1892) and Andronov (1929), both of which were concerned with two-dimensional vector fields. A general theorem on the existence of Hopf bifurcation, which is valid in n dimensions, was proved by Hopf (1942).

A simple example in the two-dimensional system is

$$Dx = -y + x\big(\theta - (x^2 + y^2)\big),$$

$$Dy = x + y\big(\theta - (x^2 + y^2)\big).$$

One equilibrium is $x^* = y^* = 0$ with stability occurring for $\theta < 0$ and the instability occurring for $\theta > 0$. That equilibrium has a pair of conjugate eigenvalues $\theta + i$ and $\theta - i$. The eigenvalues become purely imaginary, when $\theta = 0$.

Barnett and He (2004) show the following method to find Hopf bifurcation. They let $p(s) = det\ (s\mathbf{I} - \mathbf{A})$ be the characteristic polynomial of $\mathbf{A}$ and write it as

$$p(s) = c_0 + c_1 s + c_2 s^2 + c_3 s^3 + \cdots + c_{n-1} s^{n-1} + s^n.$$

They construct the following $(n-1)$ by $(n-1)$ matrix

$$\mathbf{S} = \begin{bmatrix} c_0 & c_2 & \ldots & c_{n-2} & & 1 & 0 & 0 & \ldots. & 0 \\ 0 & c_0 & c_2 & \ldots. & & c_{n-2} & 1 & 0 & \ldots & 0 \\ & & & & \ldots & & & & & \\ 0 & 0 & \ldots & 0 & & c_0 & c_2 & c_4 & \ldots & 1 \\ c_1 & c_3 & \ldots & c_{n-1} & & 0 & 0 & 0 & \ldots & 0 \\ 0 & c_1 & c_3 & \ldots & & c_{n-1} & 0 & 0\ldots & & 0 \\ & & & & \ldots. & & & & & \\ & & & & \ldots & & & & & \\ 0 & 0 & \ldots & 0 & & c_1 & c_3 & \ldots & \ldots & c_{n-1} \end{bmatrix}.$$

Let $\mathbf{S}_0$ be obtained by deleting rows 1 and $\frac{n}{2}$ and columns 1 and 2, and let $\mathbf{S}_1$ be obtained by deleting rows 1 and $\frac{n}{2}$ and columns 1 and 3. The matrix $\mathbf{A}(\theta)$ has one pair of purely imaginary eigenvalues (Guckenheimer, Myers, and Sturmfels (1997)), if

$$det\ (\mathbf{S}) = 0, \qquad det\ (\mathbf{S}_0)\, det\ (\mathbf{S}_1) > 0. \tag{1.1.6}$$

If $det\ (\mathbf{S}) = 0$ and $det\ (\mathbf{S}_0)\, det\ (\mathbf{S}_1) = 0$, then $\mathbf{A}(\theta)$ may have more than one pair of purely imaginary eigenvalues. The following condition can be used to find candidates for bifurcation boundaries:

$$det\ (\mathbf{S}) = 0, \qquad det\ (\mathbf{S}_0)\, det\ (\mathbf{S}_1) \geq 0. \tag{1.1.7}$$

Since solving (1.1.7) analytically is difficult, Barnett and He (1999) apply the following numerical procedure to find bifurcation boundaries. Without loss of generality, they initially consider only two parameters θ_1 and θ_2.

Procedure (P1)

(1) For any fixed θ_1, treat θ_2 as a function of θ_1, and find the value of θ_2 satisfying the condition $h(\theta_2) = det\big(\mathbf{A}(\theta)\big) = 0$. First find the number of zeros of $h(\theta_2)$. Starting with approximations of zeros, use the following gradient algorithm to find all zeros of $h(\theta_2)$:

$$\theta_2(n+1) = \theta_2(n) - a_n h(\theta_2)\ \Big|_{\theta_2=\theta_2(n)} \tag{1.1.8}$$

where $\{a_n, n = 0,1,2 \ldots\}$ is a sequence of positive step sizes.

(2) Repeat the same procedure to find all θ_2 satisfying (1.7).

(3) Plot all the pairs (θ_1, θ_2).

(4) Check all parts of the plot to find the segments representing the bifurcation boundaries. Then parts of the curve found in step (1) are boundaries of saddle-node bifurcations. Parts of the curve found in step (2) are boundaries of Hopf bifurcations, if the required transversality conditions are satisfied.

Pioneers in studies of Hopf bifurcations in economics include Torre (1977) and Benhabib and Nishimura (1979). Torre found the appearance of a limit cycle associated with a Hopf bifurcation boundary in Keynesian systems. Benhabib and Nishimura showed that a closed invariant curve might emerge as the result of optimization in a multi-sector neoclassical optimal growth model. These studies illustrate the existence of a Hopf bifurcation boundary in an economic model results in a solution following closed curves around the stationary state. The solution paths may be stable or unstable, depending upon the side of the bifurcation boundary on which the parameter values lie. More recent studies finding Hopf bifurcation in econometric models include Barnett and He (1999, 2002, 2008), who found bifurcation boundaries of the Bergstrom-Wymer continuous-time UK model and the Leeper and Sims Euler-equations model.

1.1.3.5 Singularity-Induced Bifurcations

This section is devoted to a dramatic kind of bifurcation found by Barnett and He (2008) in the Leeper and Sims (1977) model—singularity-induced bifurcation.

Some macroeconomic models, such as the dynamic Leontief model (Luenberger and Arbel (1977)) and the Leeper and Sims (1994) model, have the form

$$\mathbf{B}\mathbf{x}(t+1) = \mathbf{A}\mathbf{x}(t) + \mathbf{f}(t). \tag{1.1.9}$$

Here $\mathbf{x}(t)$ is the state vector, $\mathbf{f}(t)$ is the vector of driving variables, t is time, and **B** and **A** are constant matrices of appropriate dimensions. If $\mathbf{f}(t) = \mathbf{0}$, the system (1.1.9) is in the class of autonomous systems. Barnett and He (2006b) illustrate only the autonomous cases of (1.1.9).

If **B** is invertible, then we can invert **B** to acquire

$$\mathbf{x}(t+1) = \mathbf{B}^{-1}\mathbf{A}\mathbf{x}(t) + \mathbf{B}^{-1}\mathbf{f}(t),$$

so that

$$\mathbf{x}(t+1) - \mathbf{x}(t) = \mathbf{B}^{-1}\mathbf{A}\mathbf{x}(t) - \mathbf{x}(t) + \mathbf{B}^{-1}\mathbf{f}(t)$$

$$= (\mathbf{B}^{-1}\mathbf{A} - \mathbf{I})\mathbf{x}(t) + \mathbf{B}^{-1}\mathbf{f}(t),$$

which is in the form of (1.1.1).

The case in which the matrix $\mathbf{B}$ is singular is of particular interest. Barnett and He (2006b) rewrite (1.1.9) by generalizing the model to permit nonlinearity as follows:

$$\mathbf{B}(\mathbf{x}(t), \boldsymbol{\theta})\mathbf{D}\mathbf{x} = \mathbf{F}(\mathbf{x}(t), \mathbf{f}(t), \boldsymbol{\theta}). \tag{1.1.10}$$

Here $\mathbf{f}(t)$ is the vector of driving variables, and t is time. Barnett and He (2006b) consider the autonomous cases in which $\mathbf{f}(t) = \mathbf{0}$.

Singularity-induced bifurcation occurs, when the rank of $\mathbf{B}(\mathbf{x}, \boldsymbol{\theta})$ changes, as from an invertible matrix to a singular one. Therefore, the matrix must depend on $\boldsymbol{\theta}$ for such changes to occur. If the rank of $\mathbf{B}(\mathbf{x}, \boldsymbol{\theta})$ does not change according to the change of $\boldsymbol{\theta}$, then singularity of $\mathbf{B}(\mathbf{x}, \boldsymbol{\theta})$ is not sufficient for (1.1.10) to be able to produce singularity bifurcation.

Barnett and He (2006b) consider the two-dimensional state-space case and perform an appropriate coordinate transformation allowing (1.1.10) to become the following equivalent form:

$$\mathrm{B}_1(x_1, x_2, \boldsymbol{\theta})\mathrm{Dx}_1 = \mathrm{F}_1(x_1, x_2, \boldsymbol{\theta}),$$

$$0 = \mathrm{F}_2(x_1, x_2, \boldsymbol{\theta}).$$

They provide four examples to demonstrate the complexity of bifurcation behaviors that can be produced from system (1.1.10). The first two examples do not produce singularity bifurcations, since **B** does not depend on $\boldsymbol{\theta}$. In the second two examples, Barnett and He (2008) find singularity bifurcation, since **B** does depend on $\boldsymbol{\theta}$.

Example 1.1. Consider the following system modified from system (1.1.5), which has been shown to produce transcritical bifurcation:

$$Dx = \theta x - x^2, \tag{1.1.11}$$

$$0 = x - y^2. \tag{1.1.12}$$

Comparing with the general form of (1.1.10), observe that

$$\mathbf{B} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix},$$

which is singular but does not depend upon the value of θ.

The equilibria are $(x^*, y^*) = (0,0)$ and $(\theta, \pm\sqrt{\theta})$. Near the equilibrium $(x^*, y^*) = (0,0)$, the system ((1.1.11), (1.1.12)) is stable for $\theta < 0$ and unstable for $\theta > 0$. The equilibria $(x^*, y^*) = (\theta, \pm\sqrt{\theta})$ are undefined, when $\theta < 0$, and stable when $\theta > 0$. The bifurcation point is $(x, y, \theta) = (0,0,0)$. Notice before and after bifurcation, the number of differential equations and the number of algebraic equations remain unchanged. This implies that the bifurcation point does not produce singularity bifurcation, since **B** does not depend upon θ.

Example 1.2. Consider the following system modified from system (1.1.7), which can produce saddle-node bifurcation:

$$Dx = \theta - x^2, \tag{1.1.13}$$

$$0 = x - y^2. \tag{1.1.14}$$

Comparing with the general form of (1.1.10), observe that

$$\mathbf{B} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix},$$

which is singular but does not depend upon the value of θ.

The equilibria are at $(x^*, y^*) = (\sqrt{\theta}, \pm\sqrt[4]{\theta})$, defined only for $\theta \geq 0$. The system ((1.1.13), (1.1.14)) is stable around both of the equilibria $(x^*, y^*) = (\sqrt{\theta}, \pm\sqrt[4]{\theta})$ and $(x^*, y^*) = (\sqrt{\theta}, \pm\sqrt[4]{\theta})$. The bifurcation point is $(x^*, y^*, \theta) = (0,0,0)$. The three-dimensional bifurcation diagram in Barnett and He (2006b) shows that there is no discontinuity or change in dimension at the origin at the origin. The bifurcation point does not produce singularity bifurcation, since the dimension of the state space dynamics remains unchanged on either side of the origin.

Example 1.3. Consider the following system:

$$Dx = ax - x^2, \text{ with } a > 0, \tag{1.1.15}$$

$$\theta Dy = x - y^2. \tag{1.1.16}$$

Comparing with the general form of (1.1.10), observe that

$$\mathbf{B} = \begin{bmatrix} 1 & 0 \\ 0 & \theta \end{bmatrix},$$

which does depend upon the parameter θ.

When $\theta = 0$, the system has one differential equation (1.1.15) and one algebraic equation (1.1.16). If $\theta \neq 0$, the system has two differential equations (1.1.15) and (1.1.16) with no algebraic equations for nonzero θ.

The equilibria are $(x^*, y^*) = (0,0)$ and $(a, \pm\sqrt{a})$. For any value of θ, the system ((1.1.15), (1.1.16)) is unstable around the equilibrium at $(x^*, y^*) = (0,0)$. The equilibrium $(x^*, y^*) = (a, \sqrt{a})$ is unstable for $\theta < 0$ and stable for $\theta > 0$. The equilibrium $(x^*, y^*) = (a, -\sqrt{a})$ is unstable for $\theta > 0$ and stable for $\theta < 0$.

Without loss of generality, Barnett and He (2006b) normalize a to be 1. When $\theta = 0$, the system's behavior degenerates into movement along the one-dimensional curve $x - y^2 = 0$. When $\theta \neq 0$, the dynamics of the system move throughout the two-dimensional state space. The singularity bifurcation caused by the transition from nonzero θ to zero results in the drop in the dimension.

Barnett and He (2006b) observe that even if singularity bifurcation does not cause a change of the system between stability and instability, dynamical properties produced by singularity bifurcation can change. For example, if θ changes from positive to zero, when (x, y) is at the equilibrium (1,1), the system will remain stable; if θ changes from positive to zero, when (x, y) is at the equilibrium (0,0), the system will remain unstable; if θ changes from positive to zero, when(x, y) is at the equilibrium (1,-1), the system will change from unstable to stable. But in all of these cases, the nature of the disequilibrium dynamics changes dramatically, even if there is no transition between stability and instability.

Example 1.4. Consider the following system:

$$Dx = ax - x^2, \text{ with } a > 0, \tag{1.1.17}$$

$$\theta Dy = x - y. \tag{1.1.18}$$

Comparing with the general form of (1.1.10), observe that

$$\mathbf{B} = \begin{bmatrix} 1 & 0 \\ 0 & \theta \end{bmatrix}.$$

The equilibria are $(x^*, y^*) = (0,0)$ and (a, a). The system is unstable around the equilibrium $(x^*, y^*) = (0,0)$ for any value of θ. The equilibrium $(x^*, y^*) = (a, a)$ is unstable for $\theta < 0$ and stable for $\theta \geq 0$. When $\theta < 0$, the system is unstable everywhere. When $\theta = 0$, equation (1.1.18) becomes the algebraic constraint $y = x$, which is a one-dimensional ray through the origin. However, when $\theta \neq 0$, the system moves into the two-dimensional space. Even though the dimension can drop from singular bifurcation, there could be no change between stability and instability. For example, (0,0) remains unstable and (1, 1) remains stable, when $\theta \neq 0$ and $\theta = 0$.

Barnett and He (2006b) also observe that the nature of the dynamics with θ small and positive is very different from the dynamics with θ small and negative. In particular, the equilibrium at $(x^*, y^*) = (1,1)$ is stable in the former case and unstable in the latter case. Hence there is little robustness of dynamical inference to small changes of θ close to the bifurcation boundary. Barnett and Binner (2004, part 4) further investigate the subject of robustness of inferences in dynamic models.

Example 1.5. Consider the following system:

$$Dx_1 = x_3,$$

$$Dx_2 = -x_2,$$

$$0 = x_1 + x_2 + \theta x_3, \quad (1.1.19)$$

with singular matrix

$$\mathbf{B} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad (1.1.20)$$

where $\mathbf{Dx} = (Dx_1, Dx_2, Dx_3)'$.

The only equilibrium is at $\mathbf{x}^* = (x_1^*, x_2^*, x_3^*) = (0,0,0)$. For any $\theta \neq 0$, Barnett and He (2006b) solve the last equation for x_3 and substitute into the first equation to derive the following two equation system:

$$Dx_1 = -\frac{x_1 + x_2}{\theta}, \quad (1.1.21)$$

$$Dx_2 = -x_2.$$

In this case, the matrix **B** becomes the identity matrix.

This two-dimensional system is stable at $\mathbf{x}^* = (x_1^*, x_2^*) = (0,0)$ for $\theta > 0$ and unstable for $\theta < 0$. However, setting $\theta = 0$, Barnett and He (2006b) find that system (1.1.19) becomes

$$x_1 = -x_2,$$

$$Dx_2 = -x_2,$$

$$x_3 = x_2, \quad (1.1.22)$$

for all $t > 0$. This system has the following singular matrix :

$$\mathbf{B} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{1.1.23}$$

The dimension of system (1.1.22) is very different from that of (1.1.21). In system (1.1.22), there are two algebraic constraints and one differential equation, while system (1.1.21) has two differential equations and no algebraic constraints. Clearly the matrix **B** is different in the two cases with different ranks. This example shows that singular bifurcation can results from the dependence of **B** upon the parameters, even if there does not exist a direct closed-form algebraic representation of the dependence.

Barnett and He (2008) find singularity bifurcation in their research on the Leeper and Sims' Euler-equations macroeconometric model, as surveyed in section 1.3. Singularity bifurcations could similarly damage robustness of dynamic inferences with other modern Euler-equations macroeconometric models. Examples above show that implicit function systems (1.1.9) and (1.1.10) could produce singular bifurcation, while closed form differential equations systems are less likely to produce singularity bifurcation. Since Euler equation systems are in implicit function form and rarely can be solved for closed form representations, Barnett and He (2006b) conclude that singularity bifurcation should be a serious concern with modern Euler equations models.

1.2 Bergstrom—Wymer Continuous Time UK Model[3]

1.2.1. Introduction

Among the models that have direct relevance to this research are the high dimensional continuous time macroeconometric models in Bergstrom, Nowman and Wymer (1992),

[3] This section is based on Barnett and He (1999,2001b,2002).

Bergstrom, Norman, and Wandasiewicz (1994), Bergstrom and Wymer (1976), Grandmont (1998), Leeper and Sims (1994), Powell and Murphy (1997), and Kim (2000). Surveys of macroeconometric models are available in Bergstrom (1996) and in several textbooks such as Gandolfo (1996) and Medio (1992). The general theory of economic dynamics is provided, for example, in Boldrin and Woodford (1990) and Gandolfo (1992). Various bifurcation phenomena are reported in Bala (1997), Benhabib (1979), Medio (1992), Gandolfo (1992), and Nishimura and Takahashi (1992). Focused studies of stability are conducted in Grandmont (1998), Scarf (1960), and Nieuwenhuis and Schoonbeek (1997). Barnett and Chen (1988) discovered chaotic behaviors in economics. Bergstrom, Nowman, and Wandasiewicz (1994) investigate stabilization of macroeconomic models using policy control. Wymer (1997) describes several mathematical frameworks for the study of structural properties of macroeconometric models.

In section 1.2, we discuss several papers by Barnett and He on bifurcation analysis using Bergstrom, Nowman, and Wymer's continuous-time dynamic macroeconometric model of the UK economy. Barnett and He chose this policy-relevant model as their first to try, because the model is particularly well suited to these experiments. The model contains adjustment speeds producing Keynesian rigidities and hence possible Pareto improving policy intervention. In addition, as a system of second order differential equations, the model can produce interesting dynamics and possesses enough equations and parameters to be fitted plausibly to the UK data.

Barnett and He (1999) discovered that both saddle-node bifurcations and Hopf bifurcations coexist within the model's region of plausible parameter setting. Bifurcation boundaries are located and drawn. The model's Hopf bifurcation helps to provide explanations for some cyclical phenomena in the UK macroeconomy. The Barnett and He paper designed a numerical algorithm for locating the model's bifurcation boundaries. That algorithm was provide above in section 1.1.3.4.

Barnett and He (1999) observed that stability of the model had not previously been tested. They found that the point estimates of the model's parameters are outside the stable subset of the parameter space, but close enough to the bifurcation boundary so that the hypothesis of stability cannot be rejected. Confidence regions around the parameter estimates are intersected by the boundary separating stability from instability, with the point estimates being on the unstable side.

Barnett and He (2002) explored the problem of selection of a "stabilization policy." The purpose of the policy was to bifurcate the system from an unstable to a stable operating regime by moving the parameters' point estimates into the stable region. The relevant parameter space is the augmented parameter space, including both the private sector's parameters and the parameters of the policy rule. Barnett and He found that policies producing successful bifurcation to stability are difficult to determine, and the policies recommended by the originators of the model, based on reasonable economic intuition and full knowledge of their own model, tend to be counterproductive, since such policies contract the size of the stable subset of the parameter space and move that set farther away from the private sector's parameter estimates. These results point towards the difficulty of designing successful countercyclical stabilization policy in the real world, where the structure of the economy is not accurately known. Barnett and He (1999) also proposed a new formula for determining the bifurcation boundaries for transcritical bifurcations.

1.2.2. The Model[4]

The Bergstrom, Nowman, and Wymer (1992) model is described by the following 14 second-order differential equations.

$$D^2 \log C = \gamma_1(\lambda_1 + \lambda_2 - D \log C) + \gamma_2 \log \left[\frac{\beta_1 e^{-\{\beta_2(r - D\log p) + \beta_3 D \log p\}}(Q+P)}{T_1 C} \right] \quad (1.2.1)$$

$$D^2 \log L = \gamma_3(\lambda_2 - D\log L) + \gamma_4 \log\left[\frac{\beta_4 e^{-\lambda_1 t}\{Q^{-\beta_6} - \beta_5 K^{-\beta_6}\}^{-1/\beta_6}}{L} \right] \quad (1.2.2)$$

$$D^2 \log K = \gamma_3(\lambda_1 + \lambda_2 - D \log K) + \gamma_6 \log \left[\frac{\beta_5 \left(\frac{Q}{K} \right)^{1+\beta_6}}{r - \beta_7 D \log p + \beta_8} \right] \quad (1.2.3)$$

$$D^2 \log Q = \gamma_7(\lambda_1 + \lambda_2 - D \log Q) + \gamma_8 \log \left[\frac{\left\{ 1 - \beta_9 \left(\frac{qp}{p_i} \right)^{\beta_{10}} \right\} (C + G_C + DK + E_n + E_o)}{Q} \right]$$

(1.2.4)

$$D^2 \log p = \gamma_9 \left(D \log\left(\frac{w}{p} \right) - \lambda_1 \right) + \gamma_{10} \log \left[\frac{\beta_{11}\beta_4 T_2 w e^{-\lambda_1 t} \left\{ 1 - \beta_5 \left(\frac{Q}{K} \right)^{\beta_6} \right\}^{-\frac{1+\beta_6}{\beta_6}}}{p} \right] \quad (1.2.5)$$

[4] The model description is modified from Barnett and He (1999).

$$D^2 \log w = \gamma_{11}\left(\lambda_1 - D\log\left(\frac{w}{p}\right)\right) + \gamma_{12}\, D\log\left(\frac{p_i}{qp}\right)$$

$$+\gamma_{13}\log\left[\frac{\beta_4 e^{-\lambda_1 t}\left\{Q^{-\beta_6} - \beta_5 K^{-\beta_6}\right\}^{-\frac{1}{\beta_6}}}{\beta_{12} e^{\lambda_2 t}}\right] \tag{1.2.6}$$

$$D^2 r = -\gamma_{14} Dr + \gamma_{15}\left[\beta_{13} + r_f - \beta_{14} D\log q + \beta_{15}\frac{p(Q+P)}{M} - r\right] \tag{1.2.7}$$

$$D^2 \log I = \gamma_{16}\left(\lambda_1 + \lambda_2 - D\log\left(\frac{p_i I}{qp}\right)\right)$$

$$+\gamma_{17}\log\left[\frac{\beta_9\left(\frac{qp}{p_i}\right)^{\beta_{10}}\left(C + G_C + DK + E_n + E_o\right)}{\left(\frac{p_i}{qp} I\right)}\right] \tag{1.2.8}$$

$$D^2 \log E_n = \gamma_{18}(\lambda_1 + \lambda_2 - D\log E_n) + \gamma_{19}\log\left[\frac{\beta_{16} Y_f^{\beta_{17}}(p_f / qp)^{\beta_{18}}}{E_n}\right] \tag{1.2.9}$$

$$D^2 F = -\gamma_{20} DF + \gamma_{21}[\beta_{19}(Q+P) - F] \tag{1.2.10}$$

$$D^2 P = -\gamma_{22} DP + \gamma_{23}\{[\beta_{20} + \beta_{21}(r_f - D\log p_f)]K_a - P\} \tag{1.2.11}$$

$$D^2 K_a = -\gamma_{24} DK_a + \gamma_{25}\{[\beta_{22} + \beta_{23}(r_f - r) - \beta_{24} D\log q - \beta_{25} d_x](Q+P) - K_a\}$$

(1.2.12)

$$D^2 \log M = \gamma_{26}(\lambda_3 - D\log M) + \gamma_{27}\log\left[\frac{\beta_{26} e^{\lambda_3 t}}{M}\right]$$

$$+\gamma_{28}D\log\left[\frac{E_n+E_o+P-F}{(p_i/qp)I}\right]+\gamma_{29}\log\left[\frac{E_n+E_o+P-F-DK_a}{(p_i/qp)I}\right] \qquad (1.2.13)$$

$$D^2\log q=\gamma_{30}D\log(p_f/qp)+\gamma_{31}\log\left[\frac{\beta_{27}p_f}{qp}\right]+\gamma_{32}D\log\left[\frac{E_n+E_o+P-F}{(p_i/qp)I}\right]$$

$$+\gamma_{33}\log\left[\frac{E_n+E_o+P-F-DK_a}{(p_i/qp)I}\right] \qquad (1.2.14)$$

where t is time, D is the derivative operator, $Dx = dx/dt, D^2x = d^2x/dt^2,$ and $C, E_n, F, I, K, K_a, L, M, P, Q, q, r, w$ are endogenous variables whose definitions are listed below:

C = real private consumption,

E_n = real non-oil exports,

F = real current transfers abroad,

I = volume of imports,

K = amount of fixed capital,

K_a = cumulative net real investment abroad (excluding changes in official reserve),

L = employment,

M = money supply,

P = real profits, interest and dividends from abroad,

p = price level,

Q = real net output,

q = exchange rate (price of sterling in foreign currency),

r = interest rate,

w = wage rate.

The variables $d_x, E_o, G_c, p_f, p_i, r_f, T_1, T_2, Y_f$ are exogenous variables with the following definitions:

d_x = dummy variables for exchange controls ($d_x = 1$ for 1974-79, $d_x = 0$ for 1980 onwards),

E_o = real oil exports,

G_c = real government consumption,

p_f = price level in leading foreign industrial countries,

p_i = price of imports (in foreign currency),

r_f = foreign interest rate,

T_1 = total taxation policy variable, so $(Q+P)/T_1$ is real private disposable income

T_2 = indirect taxation policy variable so Q/T_2 is real output at factor cost

Y_f = real income of leading foreign industrial countries.

According to Barnett and He (1999), the structural parameters $\beta_i, i = 1,2,\ldots,27,\ \gamma_j, j = 1,2,\ldots,33,$ and $\lambda_k, k = 1,2,3,$ can be estimated from historical data. A set of their estimates using quarterly data from 1974 to 1984 are given in Table 2 of

Bergstrom, Nowman, and Wymer (1992) and the interpretations of those 14 equations are also available in Bergstrom, Nowman and Wymer (1992).

The exogenous variables satisfy the following conditions in equilibrium:

$$d_x = 0,$$

$$E_o = 0,$$

$$G_c = g^*(Q + P),$$

$$p_f = p_f^* e^{\lambda_4 t},$$

$$p_i = p_i^* e^{\lambda_4 t},$$

$$r_f = r_f^*,$$

$$T_1 = T_1^*,$$

$$T_2 = T_2^*,$$

$$Y_f = Y_f^* e^{\left(\frac{\lambda_1+\lambda_2}{\beta_{17}}\right)t} ,$$

where $g^*, p_f^*,\ p_i^*,\ r_f^*,\ T_1^*,\ T_2^*,\ Y_f^*,$ and λ_4 are constants. It has been proven that $C(t), \dots, q(t)$ in (1.2.1)-(1.2.14) change at constant rates in equilibrium. To study the dynamics of the system around the equilibrium, Barnett and He (2002) make a transformation by defining a set of new variables $y_1(t), y_2(t), \dots, y_{14}(t)$ as follows:

$$y_1(t) = \log\{C(t)/C^* e^{(\lambda_1+\lambda_2)t}\} ,$$

$$y_2(t) = \log\{ L(t)/L^* e^{\lambda_2 t}\} ,$$

$y_3(t) = \log\{K(t)/K^* e^{(\lambda_1+\lambda_2)t}\}$,

$y_4(t) = \log\{Q(t)/Q^* e^{(\lambda_1+\lambda_2)t}\}$,

$y_5(t) = \log\{p(t)/p^* e^{(\lambda_3-\lambda_1-\lambda_2)t}\}$,

$y_6(t) = \log\{w(t)/w^* e^{(\lambda_3-\lambda_2)t}\}$,

$y_7(t) = r(t) - r^*$,

$y_8(t) = \log\{I(t)/I^* e^{(\lambda_1+\lambda_2)t}\}$,

$y_9(t) = \log\{E_n(t)/E_n^* e^{(\lambda_1+\lambda_2)t}\}$,

$y_{10}(t) = \log\{F(t)/F^* e^{(\lambda_1+\lambda_2)t}\}$,

$y_{11}(t) = \log\{P(t)/P^* e^{(\lambda_1+\lambda_2)t}\}$,

$y_{12}(t) = \log\{K_a(t)/K_a^* e^{(\lambda_1+\lambda_2)t}\}$,

$y_{13}(t) = \log\{M(t)/M^* e^{\lambda_3 t}\}$,

$y_{14}(t) = \log\{q(t)/q^* e^{(\lambda_1+\lambda_2+\lambda_4-\lambda_3)t}\}$,

where C^*, $L^*, K^*, Q^*, p^*, w^*, r^*, I^*, E_n^*, F^*, P^*, K_a^*, M^*, q^*$ are functions of the vector (β, γ, λ) of 63 parameters in equations (1.2.1)-(1.2.14) and the additional parameters $g^*, p_f^*, p_i^*, r_f^*, T_1^*, T_2^*, Y_f^*, \lambda_4$.

The following is a set of differential equations derived from (1.2.1)-(1.2.14):

$$D^2 y_1 = -\gamma_1 D y_1 + \gamma_2\{\log(Q^* e^{y_4} + P^* e^{y_{11}}) - \log(Q^* + P^*) - \beta_2 y_7 + (\beta_2 - \beta_3) D y_5 - y_1\} \qquad (1.2.15)$$

$$D^2 y_2 = -\gamma_3 D y_2 + \gamma_4 \left\{ \frac{1}{\beta_6} \log\left[\frac{(Q^*)^{-\beta_6} - \beta_5 (K^*)^{-\beta_6}}{(Q^*)^{-\beta_6} e^{-\beta_6 y_4} - \beta_5 (K^*)^{-\beta_6} e^{-\beta_6 y_3}} \right] - y_2 \right\} \tag{1.2.16}$$

$$D^2 y_3 = -\gamma_5 D y_3 + \gamma_6 \{ (1+\beta_6)(y_4 - y_3) + \log[r^* - \beta_7(\lambda_3 - \lambda_1 - \lambda_2) + \beta_8]$$

$$- \log[y_7 + r^* - \beta_7 (D y_5 + \lambda_3 - \lambda_1 - \lambda_2) + \beta_8] \} \tag{1.2.17}$$

$$D^2 y_4 = -\gamma_7 D y_4 + \gamma_8 \left\{ \log\left[\frac{1 - \beta_9 (q^* p^* / p_i^*)^{\beta_{10}} e^{\beta_{10}(y_5 + y_{14})}}{1 - \beta_9 (q^* p^* / p_i^*)^{\beta_{10}}} \right] \right.$$

$$+ \log(C^* e^{y_1} + g^* (Q^* e^{y_4} + P^* e^{y_{11}}) + K^* e^{y_3} (D y_3 + \lambda_1 + \lambda_2) + E_n^* e^{y_9})$$

$$\left. - \log(C^* + g^* (Q^* + P^*) + K^* (\lambda_1 + \lambda_2) + E_n^*) - y_4 \right\} \tag{1.2.18}$$

$$D^2 y_5 = \gamma_9 (D y_6 - D y_5) + \gamma_{10} \{ y_6 - y_5 - \frac{1+\beta_6}{\beta_6} \log\left[1 - \beta_5 \left(\frac{Q^*}{K^*} \right)^{\beta_6} e^{\beta_6 (y_4 - y_3)} \right]$$

$$\left. + \frac{1+\beta_6}{\beta_6} \log\left[1 - \beta_5 \left(\frac{Q^*}{K^*} \right)^{\beta_6} \right] \right\} \tag{1.2.19}$$

$$D^2 y_6 = \gamma_{11} (D y_5 - D y_6) - \gamma_{12} (D y_5 + D y_{14}) + \gamma_{13} \left\{ \frac{1}{\beta_6} \log\left[(Q^*)^{-\beta_6} - \beta_5 (K^*)^{-\beta_6} \right] \right.$$

$$\left. - \frac{1}{\beta_6} \log\left[(Q^*)^{-\beta_6} e^{-\beta_6 y_4} - \beta_5 (K^*)^{-\beta_6} e^{-\beta_6 y_3} \right] \right\} \tag{1.2.20}$$

$$D^2 y_7 = -\gamma_{14} D y_7 + \gamma_{15} [\beta_{15} \frac{p^* e^{y_5} (Q^* e^{y_4} + P^* e^{y_{11}})}{M^* e^{y_{13}}} - \beta_{15} \frac{p^* (Q^* + P^*)}{M^*} - \beta_{14} D y_{14} - y_7] \tag{1.2.21}$$

$$D^2 y_8 = \gamma_{16} (D y_5 + D y_{14} - D y_8) + \gamma_{17} \{ (1 + \beta_{10})(y_5 + y_{14}) - y_8$$

$$+ \log [C^* e^{y_1} + g^* (Q^* e^{y_4} + P^* e^{y_{11}}) + K^* e^{y_3} (D y_3 + \lambda_1 + \lambda_2) + E_n^* e^{y_9}]$$

$$-\log[C^* + g^*(Q^* + P^*) + K^*(\lambda_1 + \lambda_2) + E_n^*]\} \tag{1.2.22}$$

$$D^2 y_9 = -\gamma_{18} D y_9 - \gamma_{19}\{\beta_{18}(y_5 + y_{14}) + y_9\} \tag{1.2.23}$$

$$D^2 y_{10} = -\{\gamma_{20} + 2(\lambda_1 + \lambda_2)\} D y_{10} - (D y_{10})^2 + \gamma_{21}\beta_{19}\left\{\frac{Q^* e^{y_4} + P^* e^{y_{11}}}{F^* e^{y_{10}}} - \frac{Q^* + P^*}{F^*}\right\} \tag{1.2.24}$$

$$D^2 y_{11} = -\{\gamma_{22} + 2(\lambda_1 + \lambda_2)\} D y_{11} - (D y_{11})^2 + \gamma_{23}\{\beta_{20} + \beta_{21}(r_f^* - \lambda_4)\}\left[\frac{K_a^* e^{y_{12}}}{P^* e^{y_{11}}} - \frac{K_a^*}{P^*}\right] \tag{1.2.25}$$

$$D^2 y_{12} = -\{\gamma_{24} + 2(\lambda_1 + \lambda_2)\} D y_{12} - (D y_{12})^2 + \gamma_{25}\{[\beta_{22} + \beta_{23}(r_f^* - r^* - y_7)$$

$$-\beta_{24}(D y_{14} + \lambda_1 + \lambda_2 + \lambda_4 - \lambda_3)]\frac{Q^* e^{y_4} + P^* e^{y_{11}}}{K_a^* e^{y_{12}}} - [\beta_{22} + \beta_{23}(r_f^* - r^*)$$

$$-\beta_{24}(\lambda_1 + \lambda_2 + \lambda_4 - \lambda_3)]\frac{Q^* + P^*}{K_a^*}\} \tag{1.2.26}$$

$$D^2 y_{13} = -\gamma_{26} D y_{13} - \gamma_{27} y_{13} + \gamma_{28}\{\frac{E_n^* e^{y_9} D y_9 + P^* e^{y_{11}} D y_{11} - F^* e^{y_{10}} D y_{10}}{E_n^* e^{y_9} + P^* e^{y_{11}} - F^* e^{y_{10}}}$$

$$+D y_5 + D y_{14} - D y_8\} + \gamma_{29}\{\log[E_n^* e^{y_9} + P^* e^{y_{11}} - F^* e^{y_{10}}$$

$$-K_a^* e^{y_{12}}(D y_{12} + \lambda_1 + \lambda_2)] - \log[E_n^* + P^* - F^* - K_a^*(\lambda_1 + \lambda_2)]$$

$$+y_5 + y_{14} - y_8\} \tag{1.2.27}$$

$$D^2 y_{14} = -\gamma_{30}(D y_5 + D y_{14}) - \gamma_{31}(y_5 + y_{14})$$

$$+\gamma_{32}\{\frac{E_n^* e^{y_9} D y_9 + P^* e^{y_{11}} D y_{11} - F^* e^{y_{10}} D y_{10}}{E_n^* e^{y_9} + P^* e^{y_{11}} - F^* e^{y_{10}}} + D y_5 + D y_{14} - D y_8\}$$

$$+\gamma_{33}\{\log[E_n^* e^{y_9} + P^* e^{y_{11}} - F^* e^{y_{10}} - K_a^* e^{y_{12}}(D y_{12} + \lambda_1 + \lambda_2)]$$

$$-\log[E_n^* + P^* - F^* - K_a^*(\lambda_1 + \lambda_2)] + y_5 + y_{14} - y_8\} \tag{1.2.28}$$

The equilibrium of the original system (1.2.1)-(1.2.14) corresponds to the equilibrium $y_i = 0, i = 1,2, \dots, 14$ of (1.2.15)-(1.2.18). The major advantage of the new system ((1.2.15)-(1.2.18)) described by (1.2.15)-(1.2.18) is that it is autonomous, but still retains all the dynamic properties of the original system (1.2.1)-(1.2.14). In Barnett and He (1999), the paper analyzes the local dynamics of (1.2.15)-(1.2.28) in a local neighborhood of the equilibrium, $y_i = 0, i = 1, 2, \dots, 14$. For simplicity, the system (1.2.15)-(1.2.28) is denoted as

$$\mathbf{Dx} = \mathbf{f}(\mathbf{x}, \boldsymbol{\theta}), \tag{1.2.29}$$

where

$$\mathbf{x} = [y_1 \;\; Dy_1 \;\; y_2 \;\; Dy_2 \dots \;\; y_{14} \;\; Dy_{14}]' \in R^{28}$$

is the state vector, while

$$\boldsymbol{\theta} = [\beta_1, \dots, \beta_{27}, \gamma_1, \dots, \gamma_{33}, \lambda_1, \lambda_2, \lambda_3]' \in R^{63}$$

is the parameter vector, and $\mathbf{f}(\mathbf{x}, \boldsymbol{\theta})$ is a vector of smooth functions of $\mathbf{x}$ and $\boldsymbol{\theta}$ obtained from (1.2.15)-(1.2.28). Note that (1.2.29) is a first-order system. The point $\mathbf{x}^* = \mathbf{0}$ is an equilibrium of (1.2.29). Let Θ denote the feasible region determined by those bounds.

1.2.3. Stability of the Equilibrium

In section 1.1.2, the discussion on stability describes a means to analyze local stability of the system through linearization. The linearized system of (1.2.15)-(1.2.28) is

$$D^2 y_1 = -\gamma_1 D y_1 + \gamma_2 \left\{ \frac{Q^* e^{y_4} + P^* e^{y_{11}}}{Q^* + P^*} - \beta_2 y_7 + (\beta_2 - \beta_3) D y_5 - y_1 \right\} \tag{1.2.30}$$

$$D^2 y_2 = -\gamma_3 D y_2 + \gamma_4 \left\{ \frac{(Q^*)^{-\beta_6} y_4 - \beta_5 (K^*)^{-\beta_6} y_3}{(Q^*)^{-\beta_6} - \beta_5 (K^*)^{-\beta_6}} - y_2 \right\} \tag{1.2.31}$$

$$D^2 y_3 = -\gamma_5 Dy_3 + \gamma_6 \{(1+\beta_6)(y_4 - y_3) - \frac{y_7 - \beta_7 Dy_5}{r^* - \beta_7(\lambda_3 - \lambda_1 - \lambda_2) + \beta_8}\} \tag{1.2.32}$$

$$D^2 y_4 = -\lambda_7 Dy_4 + \gamma_8 \{-y_4 - \frac{\beta_9 (q^* p^* / p_i^*)^{\beta_{10}}}{1 - \beta_9 (q^* p^* / p_i^*)^{\beta_{10}}} \beta_{10}(y_5 + y_{14})$$

$$+ \frac{C^* y_1 + g^*(Q^* y_4 + P^* y_{11}) + K^* Dy_3 + K^*(\lambda_1 + \lambda_2) y_3 + E_n^* y_9}{C^* + g^*(Q^* + P^*) + K^*(\lambda_1 + \lambda_2) + E_n^*} \} \tag{1.2.33}$$

$$D^2 y_5 = \gamma_9 (Dy_6 - Dy_5) + \gamma_{10} \{(1+\beta_6) \frac{\beta_5 (Q^* / K^*)^{\beta_6}}{1 - \beta_5 (Q^* / K^*)^{\beta_6}} (y_4 - y_3) + y_6 - y_5\} \tag{1.2.34}$$

$$D^2 y_6 = \gamma_{11}(Dy_5 - Dy_6) - \gamma_{12}(Dy_5 + Dy_{14}) + \gamma_{13} \frac{(Q^*)^{-\beta_6} y_4 - \beta_5 (K^*)^{-\beta_6} y_3}{(Q^*)^{-\beta_6} - \beta_5 (K^*)^{-\beta_6}} \tag{1.2.35}$$

$$D^2 y_7 = -\gamma_{14} Dy_7 + \gamma_{15} \{-\beta_{14} Dy_{14} - y_7 + \frac{\beta_{15}}{M^*}[(Q^* + P^*) p^* (y_5 - y_{13}) + p^* (Q^* y_4 + P^* y_{11})]\}$$

(1.2.36)

$$D^2 y_8 = \gamma_{16}(Dy_5 + Dy_{14} - Dy_8) + \gamma_{17}\{(1+\beta_{10})(y_5 + y_{14}) - y_8$$

$$+ \frac{C^* y_1 + g^*(Q^* y_4 + P^* y_{11}) + K^*(\lambda_1 + \lambda_2) y_3 + K^* Dy_3 + E_n^* y_9}{C^* + g^*(Q^* + P^*) + K^*(\lambda_1 + \lambda_2) + E_n^*}\} \tag{1.2.37}$$

$$D^2 y_9 = -\gamma_{18} Dy_9 - \gamma_{19}\{\beta_{18}(y_5 + y_{14}) + y_9\} \tag{1.2.38}$$

$$D^2 y_{10} = -[\gamma_{20} + 2(\lambda_1 + \lambda_2)] Dy_{10} + \frac{\gamma_{21}\beta_{19}}{F^*}[Q^*(y_4 - y_{10}) + P^*(y_{11} - y_{10})] \tag{1.2.39}$$

$$D^2 y_{11} = -[\gamma_{22} + 2(\lambda_1 + \lambda_2)] Dy_{11} + \gamma_{23}[\beta_{20} + \beta_{21}(r_f^* - \lambda_4)] \frac{K_a^*}{P^*}(y_{12} - y_{11}) \tag{1.2.40}$$

$$D^2 y_{12} = -[\gamma_{24} + 2(\lambda_1 + \lambda_2)]Dy_{12} + \gamma_{25}\{-\beta_{24}\frac{Q^* + P^*}{K_a^*}Dy_{14} - \beta_{23}\frac{Q^* + P^*}{K_a^*}y_7$$

$$+[\beta_{22} + \beta_{23}(r_f^* - r^*) - \beta_{24}(\lambda_1 + \lambda_2 + \lambda_4 - \lambda_3)]\frac{Q^*(y_4 - y_{12}) + P^*(y_{11} - y_{12})}{K_a^*}\} \quad (1.2.41)$$

$$D^2 y_{13} = -\gamma_{26} D y_{13} - \gamma_{27} y_{13}$$

$$+\gamma_{28}\left\{\frac{E_n^* Dy_9 + P^* Dy_{11} - F^* Dy_{10}}{E_n^* + P^* - F^*} + Dy_5 + Dy_{14} - Dy_8\right\}$$

$$+\gamma_{29}\left\{\frac{E_n^* y_9 + P^* y_{11} - F^* y_{10} - K_a^*(\lambda_1 + \lambda_2)y_{12} - K_a^* Dy_{12}}{E_n^* + P^* - F^* - K_a^*(\lambda_1 + \lambda_2)} + y_5 + y_{14} - y_8\right\} (1.2.42)$$

$$D^2 y_{14} = -\gamma_{30}(Dy_5 + Dy_{14}) - \gamma_{31}(y_5 + y_{14})$$

$$+\gamma_{32}\left\{\frac{E_n^* Dy_9 + P^* Dy_{11} - F^* Dy_{10}}{E_n^* + P^* - F^*} + Dy_5 + Dy_{14} - Dy_8\right\}$$

$$+\gamma_{33}\left\{\frac{E_n^* y_9 + P^* y_{11} - F^* y_{10} - K_a^*(\lambda_1 + \lambda_2)y_{12} - K_a^* Dy_{12}}{E_n^* + P^* - F^* - K_a^*(\lambda_1 + \lambda_2)} + y_5 + y_{14} - y_8\right\} \quad (1.2.43)$$

In matrix form, these equations become

$$\dot{\mathbf{x}} = \mathbf{A}(\boldsymbol{\theta})\mathbf{x}. \quad (1.2.44)$$

For the set of estimated values of $\{\beta_i\}$, $\{\gamma_j\}$, and $\{\lambda_k\}$ given in Table 2 of Bergstrom, Nowman, and Wymer (1992), all the eigenvalues of $\mathbf{A}(\boldsymbol{\theta})$ are stable, having negative real parts, except for the following three:

$$s_1 = 0.0033, \;\; s_2 = 0.009 + 0.0453\,i, \;\; s_3 = 0.009 - 0.0453i.$$

Barnett and He (1999) observe that the real parts of the unstable eigenvalues are so small and close to zero, that it is unclear whether they are caused by errors in estimation or the structural properties of the system itself.

Next, they proceed to locate the stable region and the bifurcation boundary by first looking for a stable sub-region of Θ and then expanding the sub-region to find its boundary. They first look for a parameter vector $\boldsymbol{\theta}^* \in \Theta$ such that (1.2.44) is stable. They then search for a stable region of $\boldsymbol{\theta}$ and the boundaries of bifurcation regions. To find a $\boldsymbol{\theta}^*$ such that all eigenvalues of $\mathbf{A}(\boldsymbol{\theta}^*)$ have strictly negative real parts, they first consider the following problem of minimizing the maximum real parts of eigenvalues of matrix $\mathbf{A}(\boldsymbol{\theta})$:

$$\min_{\boldsymbol{\theta}\in\Theta} R_{\max}(\mathbf{A}(\boldsymbol{\theta})) \tag{1.2.45}$$

where

$$R_{\max}\big(\mathbf{A}(\boldsymbol{\theta})\big) = \max_i\{\text{real } (\lambda_i): \lambda_1, \lambda_2, \dots, \lambda_{28} \text{ are eigenvalues of } \mathbf{A}(\boldsymbol{\theta})\}.$$

Barnett and He (1999) could not acquire a closed-form expression for $R_{\max}\big(\mathbf{A}(\boldsymbol{\theta})\big)$, since the dimension of $\mathbf{A}(\boldsymbol{\theta})$ is too high for analytic solution. Instead they employ the gradient method to solve the minimization problem (1.2.45). More precisely, let $\boldsymbol{\theta}^{(0)}$ be the estimated set of parameter values given in Table 2 of Bergstrom, Nowman, and Wymer (1992). At step $n, n \geq 0$, with $\boldsymbol{\theta}^{(n)}$, let

$$\boldsymbol{\theta}^{(n+1)} = \boldsymbol{\theta}^{(n)} - a_n \frac{\partial R_{\max}\big(\mathbf{A}(\boldsymbol{\theta})\big)}{\partial \boldsymbol{\theta}}\Big|_{\boldsymbol{\theta}=\boldsymbol{\theta}^{(n)}},$$

where $\{a_n, n = 0,1,2,\dots\}$ is a sequence of (positive) step sizes. After several iterations (20 iterations in this case), the algorithm converged to the following point, $\boldsymbol{\theta}^* \in \Theta_1$,

$$\boldsymbol{\theta}^* = [0.9400, 0.2256, 2.3894, 0.2030, 0.2603, 0.1936, 0.1829, 0.0183, 0.2470,$$
$$-0.2997, 1.0000, 23.5000, -0.0100, 0.1260, 0.0082, 13.5460, 0.4562, 1.0002,$$
$$0.0097, 0.0049, 0.2812, -0.1000, 44.9030, 0.1431, 0.0004, 71.4241, 0.8213,$$
$$3.9998, 0.8973, 0.6698, 0.0697, 0.1064, 0.0010, 3.9901, 0.3652, 1.0818,$$
$$0.0081, 3.5988, 0.6626, 0.1172, 0.8452, 0.0421, 1.4280, 0.3001, 3.9969,$$
$$3.6512, 3.9995, 4.0000, 3.9995, 3.9410, 0.5861, 0.0040, 0.7684, 0.0427,$$
$$0.1183, 0.0708, 2.3187, 0.1659, 0.0017, 0.0000, 0.0100, 0.0100, 0.0067].$$

The corresponding $R_{\max}\big(\mathbf{A}(\boldsymbol{\theta}^*)\big) = -0.0039$ implies that all eigenvalues of $\mathbf{A}(\boldsymbol{\theta}^*)$ have strictly negative real parts, and the system (1.2.44) is locally asymptotically stable around at $\boldsymbol{\theta}^*$. Barnett and He (1999) then look for the stable region of the parameter space and the bifurcation boundaries starting from this stable point.

1.2.4 Determination of Bifurcation Boundaries

The goal of this section is to find bifurcation boundaries of the model. Since the linearized system (1.2.44) only deals with local stability of the system, Barnett and He (1999) deal with local bifurcations as opposed to global bifurcations.

In the previous section, for the set of parameters given in Table 2 of Bergstrom, Nowman, and Wymer (1992), $\mathbf{A}(\boldsymbol{\theta})$ has three eigenvalues with strictly positive real parts. However, at $\boldsymbol{\theta} = \boldsymbol{\theta}^*$, found through the gradient method, all eigenvalues of $\mathbf{A}(\boldsymbol{\theta})$ have strictly negative real parts. Since eigenvalues are continuous functions of entries of $\mathbf{A}(\boldsymbol{\theta})$, there must exist at least one eigenvalue of $\mathbf{A}(\boldsymbol{\theta})$ with zero real part on the bifurcation boundary. Different types of bifurcations may occur and three types of bifurcations are discussed in Barnett and He (1999,2002): saddle-node bifurcations, Hopf bifurcations, and transcritical bifurcations.

1.2.4.1 Saddle-node and Hopf Bifurcations

In systems generated by autonomous ordinary differential equations, a saddle-node bifurcation occurs, when the critical equilibrium has a simple zero eigenvalue. If $det\big(\mathbf{A}(\boldsymbol{\theta})\big) = 0$, then $\mathbf{A}(\boldsymbol{\theta})$ has at least one zero eigenvalue. Therefore, Barnett and He (1999) start from $det\big(\mathbf{A}(\boldsymbol{\theta})\big) = 0$ to look for bifurcation boundaries. To demonstrate the feasibility of this approach, Barnett and He (1999) consider the bifurcation boundaries for β_2 and β_5. The following theorem is proved in Barnett and He (1999) as their theorem 1.

Theorem 1.2.1. The bifurcation boundary for β_2 and β_5 is determined by

$$1.36\beta_2\beta_5 + 21.78\beta_5 - 2.05\beta_2 - 10.05 = 0. \tag{1.2.46}$$

A Hopf bifurcation occurs at points at which the system has a nonhyperbolic equilibrium associated with a pair of purely imaginary, but non-zero, eigenvalues and when additional transversality conditions are satisfied. Barnett and He (1999) use the Procedure (P1) introduced in section 1.1.3.4 to find Hopf bifurcation. They numerically find boundaries of saddle-node bifurcations and Hopf bifurcations for β_2 and β_5, the surface of the bifurcation boundary for β_2, β_5 and β_{15}, Hopf bifurcation boundary for γ_8 and β_{15}, and the three dimensional Hopf bifurcation boundary for γ_8, β_{15} and β_2. Barnett and He (1999) conclude that the method is applicable to any number of parameters.

1.2.4.2 Transcritical Bifurcations

A new method of finding transcritical bifurcations is proposed in Barnett and He (2002). Again Barnett and He (2002) start from $det(\mathbf{A}(\boldsymbol{\theta})) = 0$ to look for bifurcation boundaries.

Without loss of generality, Barnett and He (2002) consider bifurcations when two parameters θ_i, θ_j change, while others are kept at $\boldsymbol{\theta}^*$. The matrix $\mathbf{A}(\boldsymbol{\theta})$ is therefore rewritten as

$$\mathbf{A}(\boldsymbol{\theta}) = \mathbf{A}(\boldsymbol{\theta}^*) + \mathbf{B}(\boldsymbol{\theta}^*)\mathbf{D}(\boldsymbol{\mu})\mathbf{C}(\boldsymbol{\theta}^*), \tag{1.2.47}$$

where $\boldsymbol{\mu} = [\theta_i, \theta_j]$, and $\mathbf{D}(\boldsymbol{\mu})$ is a matrix of appropriate dimension. The dimension of $\mathbf{D}(\boldsymbol{\mu})$ is usually much smaller than that of $\mathbf{A}(\boldsymbol{\theta})$. In this case, the following proposition, proved in Barnett and He (2002) as their Proposition 1, is useful for simplifying the calculation of transcritical bifurcation boundaries.

Proposition 1.2.1. Assume that $\mathbf{A}(\boldsymbol{\theta})$ has structure (1.2.47) and that all eigenvalues of $\mathbf{A}(\boldsymbol{\theta}^*)$ have strictly negative real parts. Then $det(\mathbf{A}(\boldsymbol{\theta})) = 0$, if and only if

$$det\left(\mathbf{I} + \mathbf{D}(\boldsymbol{\mu})\mathbf{C}(\boldsymbol{\theta}^*)\mathbf{A}^{-1}(\boldsymbol{\theta}^*)\mathbf{B}(\boldsymbol{\theta}^*)\right) = 0. \tag{1.2.48}$$

Barnett and He (2002) demonstrate the usefulness of this approach by considering the bifurcation boundary for $\boldsymbol{\mu} = [\theta_2, \theta_{23}] = [\beta_2, \beta_{23}]$. They find that only the following entries of $\mathbf{A}(\boldsymbol{\theta})$ are functions of $\boldsymbol{\mu}$:

$$a_{2,10}(\boldsymbol{\mu}) = \gamma_2(\beta_2 - \beta_3), \qquad a_{2,13}(\boldsymbol{\mu}) = -\gamma_2\beta_2\ ,$$

$$a_{24,7}(\boldsymbol{\mu}) = \frac{\gamma_{25}\delta Q^*}{K_a^*}, \qquad a_{24,13}(\boldsymbol{\mu}) = -\frac{\gamma_{25}\beta_{23}(Q^* + P^*)}{K_a^*},$$

$$a_{24,21}(\boldsymbol{\mu}) = \frac{\gamma_{25}\delta P^*}{K_a^*}, \qquad a_{24,23}(\boldsymbol{\mu}) = -\frac{\gamma_{25}\delta(Q^* + P^*)}{K_a^*}$$

where $\delta = \beta_{22} + \beta_{23}(r_f - r^*) - \beta_{24}(\lambda_1 + \lambda_2 + \lambda_4 - \lambda_3)$. In this case, $\mathbf{B}(\boldsymbol{\theta}^*) \in R^{28\times 2}$ has all zero entries except that its (2,1) entry is 1 and its (24,2) entry is 1. The matrix $\mathbf{C}(\boldsymbol{\theta}^*) \in R^{5\times 28}$ has zero entries, except the entries are 1 at the following locations: (1,7), (2,10), (3,13), (4,21), (5,23). The matrix $\mathbf{D}(\boldsymbol{\mu})$ is

$$\mathbf{D}(\boldsymbol{\mu}) = \mathbf{d}(\boldsymbol{\mu}) - \mathbf{d}(\boldsymbol{\theta}^*),$$

with

$$\mathbf{d}(\boldsymbol{\mu}) = \begin{bmatrix} 0 & a_{2,10}(\boldsymbol{\mu}) & a_{2,13}(\boldsymbol{\mu}) & 0 & 0 \\ a_{24,7}(\boldsymbol{\mu}) & 0 & a_{24,13}(\boldsymbol{\mu}) & a_{24,21}(\boldsymbol{\mu}) & a_{24,23}(\boldsymbol{\mu}) \end{bmatrix}.$$

Using Proposition 1.2.1, Barnett and He (2002) observe that $det(\mathbf{A}) = 0$ is equivalent to

$$det\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \mathbf{D}(\boldsymbol{\mu})\mathbf{C}(\boldsymbol{\theta}^*)\mathbf{A}^{-1}(\boldsymbol{\theta}^*)\mathbf{B}(\boldsymbol{\theta}^*)\right) = 0,$$

where

$$\mathbf{C}(\boldsymbol{\theta}^*)\mathbf{A}^{-1}(\boldsymbol{\theta}^*)\mathbf{B}(\boldsymbol{\theta}^*) = \begin{bmatrix} 13.7090 & -17.1187 \\ 0 & 0 \\ -1.7276 & 2.1573 \\ -616.4935 & 389.2039 \\ -616.4935 & 389.2039 \end{bmatrix}.$$

Equivalently, they obtain the bifurcation boundary:

$$-14.23 + 15.91\,\theta_2 + 0.28\,\theta_{23} - 0.50\theta_2\theta_{23} = 0.$$

When parameters take values on the bifurcation boundary, stability of the system (1.2.29) needs to be determined by examining the higher order terms in $\mathbf{Dx} = \mathbf{A}(\boldsymbol{\theta})\mathbf{x} +$

$\mathbf{F}(\mathbf{x}, \boldsymbol{\theta})$ with center manifold theory. Barnett and He (2002) write $\mathbf{Dx} = \mathbf{A}(\boldsymbol{\theta})\mathbf{x} + \mathbf{F}(\mathbf{x}, \boldsymbol{\theta})$ through appropriate coordinate transformation as (see Glendinning (1994) or Guckenheimer and Holmes (1983)):

$$Dx_1 = A_1(\boldsymbol{\theta})x_1 + F_1(x_1, x_2, \boldsymbol{\theta}), \tag{1.2.49}$$

$$Dx_2 = A_2(\boldsymbol{\theta})x_2 + F_2(x_1, x_2, \boldsymbol{\theta}), \tag{1.2.50}$$

where all eigenvalues of $A_1(\boldsymbol{\theta})$ have zero real parts and all eigenvalues of $A_2(\boldsymbol{\theta})$ have strictly negative real parts. By center manifold theory, there exists a center manifold, $x_2 = h(x_1)$, such that

$$h(0) = 0 \text{ and } Dh(0) = 0.$$

By substituting $x_2 = h(x_1)$ into (2.49), Barnett and He (2002) obtain

$$Dx_1 = A_1(\boldsymbol{\theta})x_1 + F_1(x_1, h(x_1), \boldsymbol{\theta}). \tag{1.2.51}$$

The stability of (1.2.29) is connected to that of (1.2.51) through the following theorem.

Theorem 1.2.2. (Henry (1981), Carr (1981)) If the origin of (1.2.51) is locally asymptotically stable (respectively unstable), then the origin of (1.2.29) is also locally asymptotically stable (respectively unstable).

By substituting $x_2 = h(x_1)$ into (1.2.50), Barnett and He (2002) observes that $h(x_1)$ satisfies

$$\begin{aligned} Dx_2 &= Dh(x_1)Dx_1 = Dh(x_1)[A_1(\boldsymbol{\theta})x_1 + F_1(x_1, h(x_1), \boldsymbol{\theta})] \\ &= A_2(\boldsymbol{\theta})h(x_1) + F_2(x_1, h(x_1), \boldsymbol{\theta}), \end{aligned}$$

or $h(x_1)$ satisfies

$$Dh(x_1)[A_1(\boldsymbol{\theta})x_1 + F_1(x_1, h(x_1), \boldsymbol{\theta})] = A_2(\boldsymbol{\theta})h(x_1) + F_2(x_1, h(x_1), \boldsymbol{\theta}), \qquad (1.2.52)$$

$$h(0) = 0, Dh(0) = 0. \qquad (1.2.53)$$

For most cases, especially codimension-1 bifurcations, the dimension of (1.2.51) is usually one or two. In the case of transcritical bifurcations, the dimension of (1.2.51) is one. Since solving (1.2.52) and (1.2.53) is difficult, Barnett and He (2002) use a Taylor series approximation with several terms to determine the local asymptotic stability or instability of (1.2.51). In this case, let

$$F_1(x_1, x_2, \boldsymbol{\theta}) = a_1 \frac{x_1^2}{2!} + x_1 a_2 x_2 + a_3 \frac{x_1^3}{3!} + \cdots,$$

$$F_2(x_1, x_2, \boldsymbol{\theta}) = b_1 \frac{x_1^2}{2!} + x_1 b_2 x_2 + b_3 \frac{x_1^3}{3!} + \cdots.$$

Barnett and He (2002) assume that $h(x_1)$ has the following Taylor expansion

$$h(x_1) = \alpha \frac{x_1^2}{2!} + \beta \frac{x_1^3}{3!} + \cdots.$$

Then (1.2.52) becomes

$$\left(\alpha x_1 + \beta \frac{x_1^2}{2!} + \cdots\right)\left[A_1(\boldsymbol{\theta})x_1 + a_1 \frac{x_1^2}{2!} + x_1 a_2 \left(\alpha \frac{x_1^2}{2!} + \beta \frac{x_1^3}{3!} + \cdots\right) + a_3 \frac{x_1^3}{3!} + \cdots\right]$$

$$= A_2(\boldsymbol{\theta})\left(\alpha \frac{x_1^2}{2!} + \beta \frac{x_1^3}{3!} + \cdots\right) + b_1 \frac{x_1^2}{2!} + x_1 b_2 \left(\alpha \frac{x_1^2}{2!} + \beta \frac{x_1^3}{3!} + \cdots\right) + b_3 \frac{x_1^3}{3!} + \cdots \quad .$$

By comparing coefficients of the same order terms and also observing that $A_1(\boldsymbol{\theta}) = 0$ at a bifurcation point, Barnett and He (1999) observe that

$$\alpha = -A_2^{-1}(\boldsymbol{\theta})b_1, \quad \beta = A_2^{-1}(\boldsymbol{\theta})(\alpha a_1 - b_2 \alpha).$$

Therefore, (1.2.51) becomes

$$Dx_1 = A_1(\boldsymbol{\theta})x_1 + a_1\frac{x_1^2}{2!} + \left(\frac{a_2\alpha}{2!} + \frac{a_3}{3!}\right)x_1^3 + \cdots. \tag{1.2.54}$$

The stability analysis of (1.2.54) determines the stability characteristics of $\mathbf{Dx} = \mathbf{A}(\boldsymbol{\theta})\mathbf{x} + \mathbf{F}(\mathbf{x}, \boldsymbol{\theta})$.

Without loss of generality, Barnett and He (2002) consider the stability of the system on the transcritical bifurcation boundary for parameters β_2, β_{23}. Considering the point $(\beta_2, \beta_{23}) = (0.1068, 55.9866)$ on the boundary and using previous approach, Barnett and He (1999) find that (1.2.51) becomes $Dx_1 = 0.1308\, x_1^2 + o(x_1^2)$, which is locally asymptotically unstable at $x_1 = 0$. Therefore, it follows from center manifold theory that the system (1.2.29) is locally asymptotically unstable at this transcritical bifurcation point. Furthermore, Barnett and He (2002) numerically find boundaries of both Hopf and transcritical bifurcations for θ_2 and θ_{62}, for θ_2, $\theta_{23}$and θ_{62}, for $\theta_{23}$and θ_{62} and for θ_{12}, $\theta_{23}$and θ_{62}.

1.2.5 Stabilization Policy

We have seen in the previous section that both transcritical and Hopf bifurcations exist in the UK continuous time macroeconometric model. In this section, we provide Barnett and He's (2002) results investigating the control of bifurcations using fiscal feedback laws. They define stabilization policy to be intentional movement of bifurcation regions through policy intervention, with the intent of moving the stable region to include the parameters. However, there would be no need for stabilization policy, if the parameters were inside the stable region without policy.

Barnett and He (2002) first consider the effect of a heuristically plausible fiscal policy of the following form, as suggested in Bergstrom, Nowman, and Wymer (1992):

$$D\log T_1 = \gamma\left[\beta\log\left\{\frac{Q}{Q^* e^{(\lambda_1+\lambda_2)t}}\right\} - \log\left\{\frac{T_1}{T_1^*}\right\}\right]. \tag{1.2.55}$$

The control feedback rule (1.2.55) adjusts the fiscal policy instrument, T_1, towards a partial equilibrium level, which is an increasing function of the ratio of output to its steady state level. In (1.2.55), β is a measure of the strength of the feedback, and γ governs the speed of adjustment. According to Bergstrom, Nowman, and Wymer (1992), the control law (1.2.55) can reduce the positive real parts of unstable eigenvalues through proper choices of parameters β, γ. The intent is for the policy to be stabilizing. However, Barnett and He (2002) tried the following procedure and found that the control law (1.2.55) is unlikely to stabilize the systems (1.2.1)-(1.2.14). First, they define $y_{15} = \log\left\{\frac{T_1}{T_1^*}\right\}$, and then they find that y_{15} satisfies

$$Dy_{15} = \gamma\beta y_4 - \gamma y_{15}.$$

They add this equation to the system (1.2.29) and obtain

$$\mathbf{Dw} = \mathbf{A}'(\boldsymbol{\theta})\mathbf{w} + \mathbf{F}'(\mathbf{x}, \boldsymbol{\theta}), \tag{1.2.56}$$

where

$$\mathbf{w} = \begin{bmatrix} \mathbf{x} \\ y_{15} \end{bmatrix}, \quad \mathbf{F}'(\mathbf{x}, \boldsymbol{\theta}) = \begin{bmatrix} \mathbf{F}(\mathbf{x}, \boldsymbol{\theta}) \\ 0 \end{bmatrix},$$

and $\mathbf{A}'(\boldsymbol{\theta})$ is the corresponding coefficient matrix.

They then consider three sets of parameter values: $\beta = 0.04, \gamma = 0.02;\ \beta = 0.01, \gamma = 0.05$; and $\beta = 0, \gamma = 0$. The case, $\beta = 0, \gamma = 0$, corresponds to the original system (1.2.1)-(1.2.14), in which no fiscal policy control is applied. Barnett and He (2002) illustrate the effect of a simple fiscal policy in three cases, indicating that some stable regions could be destabilized and some unstable regions could be stabilized. But since the feasible region is smaller under control than without control, Barnett and He conclude that the policy is not likely to succeed.

Barnett and He (2002) next consider a more sophisticated fiscal control policy, based upon optimum control theory, with the control being

$$u = \log\left\{\left\{\frac{T_1}{T_1^*}\right\}\right\}. \tag{1.2.57}$$

Under the control (1.2.57), the system (1.2.29) becomes

$$\mathbf{Dx} = \mathbf{A}(\boldsymbol{\theta})\mathbf{x} + \mathbf{B}u + \mathbf{F}(\mathbf{x}, \boldsymbol{\theta}), \tag{1.2.58}$$

where $\mathbf{B} = [0 \quad -\gamma_2 \quad 0 \ \dots \ 0]^T \in R^{28}$. The controllability matrix $[\mathbf{B} \quad \mathbf{AB} \dots \quad \mathbf{A}^{27}\mathbf{B}]$ has rank 7, implying that the pair $(\mathbf{A}, \mathbf{B})$ is not controllable. Therefore, it is not possible to set the closed-loop eigenvalues of the coefficient matrix of (1.2.58) arbitrarily.

Nevertheless, the numerical procedure of Khalil (1992) shows that there exists a linear transformation, $\mathbf{z} = \mathbf{Tx}$, such that

$$\mathbf{Dz} = \begin{bmatrix} \mathbf{A}_{11} & \mathbf{0} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{bmatrix}\mathbf{z} + \begin{bmatrix} \mathbf{0} \\ \mathbf{B}_2 \end{bmatrix}u,$$

where $\mathbf{A}_{11} \in R^{21\times 21}, \mathbf{A}_{21} \in R^{7\times 21}, \mathbf{A}_{22} \in R^{7\times 7}, \mathbf{B}_2 = [\,0 \dots 0\ 1] \in R^7$,

$$\mathbf{TA}(\boldsymbol{\theta})\mathbf{T}^{-1} = \begin{bmatrix} \mathbf{A}_{11} & \mathbf{0} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{bmatrix}, \quad \mathbf{TB} = \begin{bmatrix} \mathbf{0} \\ \mathbf{B}_2 \end{bmatrix},$$

and $(\mathbf{A}_{22}, \mathbf{B}_2)$ is controllable. Further, all eigenvalues of $\mathbf{A}_{11}$ have negative real parts, implying that $(\mathbf{A}(\boldsymbol{\theta}), \mathbf{B})$ is stabilizable.

To obtain a feedback control law stabilizing (1.2.58), Barnett and He (2002) consider minimizing

$$J = \int_0^\infty [\mathbf{x}^\mathrm{T}\mathbf{U}\mathbf{x} + Vu^2]dt,$$

where $\mathbf{U} \in R^{28\times 28}$ and $V \in R^1$ are positive definite. According to linear system theory, the optimal feedback control law is given by

$$u = \mathbf{Kx}, \quad \mathbf{K} = -V^{-1}\mathbf{B}^\mathrm{T}\mathbf{P},$$

where $\mathbf{P}$ is positive definite and solves the algebraic Ricatti equation $\mathbf{PA} + \mathbf{A}^\mathrm{T}\mathbf{P} - \mathbf{PB}V^{-1}\mathbf{B}^\mathrm{T}\mathbf{P} + \mathbf{U} = \mathbf{0}$. Choosing $\mathbf{U} = \mathbf{I}$ and $V = 1$, Barnett and He (2002) get

$$\mathbf{K} = [\,1.5036, 0.4754, 0.0178, 0.0307, -1.1897, 18.5851, 7.2979, 1.9063, 2.3147,$$
$$23.2392, 0.7488, 7.2091, 38.9965, 39.4000, 0.1841, 0.2129, 0.3061, 0.0494, -0.0027,$$
$$0.0000, -0.0013, -0.0002, 0.9550, 1.8482, -0.3329, -0.5475, 0.9369, -1.0402].$$

(1.2.59)

Under the control $u = \mathbf{Kx}$, equation (1.2.58) becomes

$$\mathbf{Dx} = [\mathbf{A}(\boldsymbol{\theta}) + \mathbf{BK}]\mathbf{x} + \mathbf{F}(\mathbf{x}, \boldsymbol{\theta}). \tag{1.2.60}$$

Since all the eigenvalues of $\mathbf{A} + \mathbf{BK}$ have strictly negative real parts under the choice of $\mathbf{K}$, the state feedback law $u = \mathbf{Kx}$ indeed stabilizes the system (1.2.60). Barnett and He (2002) also confirm by direct verification that there exist no bifurcations under the control law (1.2.60) for (β_2, β_5).

Barnett and He (2002) further investigate whether there is a parameter $\boldsymbol{\theta}' \in \Theta$ at which the system (1.2.60) is unstable. They check the stability of (1.2.60) under the control law (1.2.60) for all parameter $\boldsymbol{\theta} \in \Theta$. The following $\boldsymbol{\theta}' \in \Theta_1$ were found

$$\boldsymbol{\theta}' = [0.9400, 0.5074, 2.0913, 0.2030, 0.2612, 0.1933, 0.2309, 0.0000, 0.2510, -0.3423,$$
$$1.0000, 23.5000, -0.0100, 0.2086, 0.0332, 13.5460, 0.4562, 0.9322, 0.0100, 0.0034,$$
$$0.1324, -0.5006, 100.0000, 0.0000, 0.0004, 71.4241, 0.8213, 4.0000, 1.0289, 0.3631,$$
$$0.1201, 0.1000, 0.0010, 3.7015, 0.4860, 1.1270, 0.0042, 3.3994, 0.4802, 0.1300, 0.6851,$$
$$0.0620, 1.2134, 0.3830, 4.0000, 3.2535, 3.8592, 4.0000, 4.0000, 3.5723, 0.4775, 0.0071,$$
$$0.6104, 0.0143, 0.1718, 0.1227, 2.5551, 0.1833, 0.0035, 0.0000, 0.0018, 0.0004, 0.0100].$$

The corresponding $R_{\max}\left(\mathbf{A}(\boldsymbol{\theta}')\right) = 0.4971$. Hence, there indeed exists a parameter $\boldsymbol{\theta}' \in \Theta_1$ at which (1.2.60) is unstable.

Barnett and He (2002) investigate whether the use of an optimal control feedback policy with a structural model would be easily implemented, if the Lucas critique and time inconsistency issues did not exist. It is often believed that designing such active policy would be easy, if it were not for the problems produced by the Lucas critique and by the time inconsistency of optimal control. However, Barnett and He (2002) find that even without those problems, the design of a successful feedback policy can be difficult. They consider a policy to be successful, if the policy shifts the bifurcation boundaries such that the stable region moves towards the point estimates of the parameters. Then the probability is increased that the stable region will include the values of the parameters. Barnett and He (2002) find that Bergstrom's proposed selection of a fiscal policy feedback rule for his own UK model is counterproductive for three reasons: (1) the resulting policy equation derived from optimal control theory is complicated and depends heavily upon the model; (2) the problem of robustness of the optimal control policy to specification error is not addressed; and (3) the problems of possible time inconsistency of optimal control policy are not taken into consideration. The effects of policy feedback rules can depend upon the complicated geometry of bifurcation boundaries and how they are moved by augmentation of the model by the feedback rule. As a result, Barnett and He (2002) conclude that such policies can be counterproductive.

1.3 Leeper and Sims Model

1.3.1 Introduction

Barnett and He (2008) conducted a bifurcation analysis of the best-known Euler-equations general-equilibrium macroeconometric model: the Leeper and Sims (1994) model and found the existence of singularity bifurcation boundaries within the parameter space. This section surveys Barnett and He's (2008)'s bifurcation analysis of that model.

Barnett and He (2008) provided initial confirmation of Grandmont's views about bifurcation. Grandmont (1985) found that the parameter space of even the most classical dynamic general-equilibrium macroeconomic models is stratified into bifurcation regions. This result challenged the prior common view that different kinds of economic dynamics can only be attributed to different kinds of structures. But he was not able to reach conclusions about policy relevance, since his results were based on a model in which all policies are Ricardian equivalent, no frictions exist, employment is always full, competition is perfect, and all solutions are Pareto optimal. Nevertheless, robustness of dynamical inferences can be seriously damaged by the stratification of a confidence region into bifurcated subsets, when a bifurcation boundary crosses the confidence region of a parameter. Policy relevance was introduced by Barnett and He (1999, 2001a, 2002), who investigated Bergstrom-Wymer continuous-time dynamic macroeconometric model of UK economy. That Keynesian model does permit introduction of welfare improving countercyclical policy. Barnett and Duzhak (2008, 2010) further explored policy relevance by demonstrating the existence of Hopf and flip bifurcations within the more recent class of New Keynesian models.

There is a large literature on dynamic macroeconometric models.[5] In particular, the Lucas critique has motivated development of Euler-equations models with policy-invariant deep parameters. A seminal example in this class is the Leeper and Sims model, which contains parameters of consumer and firm behavior as deep parameters of tastes and technology. The deep parameters are invariant to government policy rule changes, and hence immune to the Lucas critique.[6] The dimension of the state space in the Leeper and Sims model is substantially lower than in the Bergstrom--Wymer UK model, but still too high for analysis by available analytical approaches. Through numerical procedures, Barnett and He (2008) find that the dynamics of the Leeper and Sims model are complicated by the model's Euler equations structure. The model consists of both differential equations and algebraic

[5] See Barnett and He (2008), footnote 2.

[6] Similar models are developed in Kim (2000) and in Binder and Pesaran (1999), according to Barnett and He (2008), footnote 3.

constraints. Barnett and He (2008) found that the order of the dynamics of the Leeper and Sims model could change within a small neighborhood of the estimated parameter values. Within this small neighborhood close to a bifurcation boundary, one eigenvalue of the linearized part of the model can move quickly from finite to infinite and back again to finite. Barnett and He (2008) state that a large stable eigenvalue indicates that some variables can respond rapidly to changes of other variables. A large unstable eigenvalue indicates one variable's rapid diversion away from other variables, while an infinity eigenvalue indicates existence of a pure algebraic relationships among the variables. Due to the nature of the mapping from parameter space to functional space of dynamical solutions, the sensitivity to the setting of the parameters presents serious challenges to the robustness of dynamical inferences.

Barnett and He's (2008)'s bifurcation analysis of the Leeper and Sims model not only confirm the policy relevance of Grandmont's views but also reveal the existence of a singularity bifurcation boundary within a small neighborhood of the estimated parameter values. Singularity bifurcation, surveyed in section 1, had not previously been encountered in economics, although is known in the engineering and mathematics literatures. On the singularity boundary, the number of differential equations will decrease, while the number of algebraic constraints will increase. Such change in the order of dynamics had not previously been found with macroeconometric models. Barnett and He (2008) speculate that singularity bifurcation may be a common property of Euler equations models. Even though the dimension of the dynamics can be the same on both sides of a singularity bifurcation boundary, the nature of the dynamics on one side may differ dramatically from the nature of the dynamics on the other side. Hence the implications of singularity bifurcation are not limited to the change in the dimension of the dynamics directly on the bifurcation boundary. These results cast into doubt the robustness of dynamical inferences acquired by simulation only at the point estimate of the parameters. Barnett and He (2008) advocate simulating models at various settings throughout the parameters' confidence region, rather than solely at the parameters point estimates.

Since the US data used in the model include imported and exported goods, the Leeper and Sims model, although specified as a closed economy model, is implicitly open economy. Barnett and He (2008) consider extension of their analysis to an explicitly open-economy Euler-equations model. In section 1.6, we survey research on bifurcation phenomena in explicitly open-economy New Keynesian models.

1.3.2 The Model[7]

The Leeper and Sims (1994) model includes the dynamic behavior of consumers, firms, and government. Consumers and firms maximize their respective objective functions, and the government pursues countercyclical policy objectives through monetary and tax policies satisfying an intertemporal government budget constraint. Parameters of consumer and firm behavior are the deep parameters of tastes and technology and are invariant to government policy rule changes. The model consists of both ordinary differential equations and algebraic constraints. The resulting system is called a differential/algebraic system in systems theory. The detailed derivation of the models is available in Leeper and Sims (1994) and will not be repeated in this survey.

The Leeper and Sims model consists of the following 12 state variables.

[7] The model description is modified from Barnett and He (2008).

L = labor supply,

C^* = consumption net of transaction costs,

M = consumer demand for non-interest-bearing money,

D = consumer demand for interesting-bearing money,

K = capital,

Y = factor income from capital and labor, excluding interest on government debt,

C = gross consumption,

Z = investment,

X = consumption goods aggregate price,

Q = investment goods price,

V = income velocity of money,

P = general price level.

The consumer maximizes utility according to

$$E\left[\int_0^\infty \exp\,(-\int_0^t \beta(s)ds)\frac{(C^{*\pi}(1-L)^{1-\pi})^{1-\gamma}}{1-\gamma}dt\right]$$

subject to

$$XC + QZ + \tau + \frac{\dot{M} + \dot{D}}{P} = Y + \frac{iD}{P},$$

$$XC^* + \phi VY = XC,$$

$$\dot{K} = Z - \delta K,$$

$$Y = rK + wL + S,$$

$$V = \frac{PY}{M},$$

where $\pi \in (0,1)$ and $\gamma > 0$ are parameters; $0 \leq \beta(s) \leq 1$ is the subjective rate of time preference at time $s;$ τ is the level of lump-sum taxes paid by the representative consumer; i is the nominal rate of return earned on government bonds; S is the sum of dividends received by the representative consumer, w is the wage rate; $\varphi > 0$ is the transaction cost per unit of $VY;$ $\delta \geq 0$ is the rate of depreciation of capital; and r = rental rate of return on capital. Parameters in this model are not necessarily assumed to be constant.

The firms maximize profits according to

$$\max\left\{ X(C+g) + QI^{*} + A(\alpha K^{\sigma} + L^{\sigma})^{\frac{1}{\sigma}} - rK - wL - ((C+g)^{\mu} + \theta I^{*\,\mu})^{\frac{1}{\mu}} \right\},$$

where g is the level of government purchases. The following are parameters:

$A > 0, \alpha > 0, \theta > 0, \mu \geq 0,$ and $0 \leq \sigma \leq 1.$

The market-clearing condition is $I^{*} = Z + nK$, where $n =$ the fraction of existing capital purchased by the government for distribution to the newborn. Investment goods, I^{*}, produced by the firm include both those bought by the existing population, and those purchased by the government for distribution to the newborn, as indicated by the market-clearing condition.

In this model, the state variables satisfy the following differential equations:

$$\frac{1}{p}\left(\dot{M} + \dot{D}\right) = Y - XC - QL + \frac{iD}{P} + \tau \tag{1.31}$$

$$\dot{K} = Z - \delta K \tag{1.3.2}$$

$$\left(1-\pi(1-\gamma)\right)\frac{\dot{C^{*}}}{C^{*}} + (1-\gamma)(1-\pi)\frac{\dot{L}}{1-L} + \frac{\dot{X}}{X} + \frac{\dot{P}}{P}$$

$$= i - \beta + \frac{\dot{\pi}}{\pi} + \dot{\pi}(1-\gamma)\log\left(\frac{C^*}{1-L}\right), \tag{1.3.3}$$

$$\frac{\dot{P}}{P} + \frac{\dot{Q}}{Q} = i + \delta - (1 - 2\phi V)\frac{r}{Q} \tag{1.3.4}$$

Equation (1.3.1) represents the consumers' budget constraint. Equation (1.3.2) is the law of motion for capital, and equations (1.3.3) and (1.3.4) are the first-order conditions derived from the consumers' optimization problem. In addition, the state variables also satisfy the following algebraic constraints.

$$X = \left(\frac{Y}{C+g}\right)^{1-\mu}, \tag{1.3.5}$$

$$Q = \theta\left(\frac{Y}{Z+nK}\right)^{1-\mu}, \tag{1.3.6}$$

$$r = A^{\sigma}\alpha\left(\frac{Y}{K}\right)^{1-\delta}, \tag{1.3.7}$$

$$w = A^{\sigma}\left(\frac{Y}{L}\right)^{1-\delta}, \tag{1.3.8}$$

$$XC^* + \phi VY = XC, \tag{1.3.9}$$

$$Y = rK + wL + S, \tag{1.3.10}$$

$$V = \frac{PY}{M}, \tag{1.3.11}$$

$$X(C+g) + Q(Z+nK) = Y, \tag{1.3.12}$$

$$(1-2\phi V)\frac{w}{X}=\frac{1-\pi}{\pi}\frac{C^*}{1-L}, \tag{1.3.13}$$

$$i=\phi V^2. \tag{1.3.14}$$

Equations (1.3.5)-(1.3.8) are obtained from the first-order conditions of the firms' optimization problem. Equation (1.3.9) defines consumption net of transaction costs, with total output serving as a measure of the level of transactions at a given point in time. Equation (1.3.10) defines income. Equation (1.3.11) is the income velocity of money. Equation (1.3.12) is the social resources constraint. Equations (1.3.13)-(1.3.14) are obtained from the first-order conditions for the consumers' optimization problem.

The control variables consist of the nominal rate of return on government bonds, i, and the level of lump-sum taxes, τ. According to Barnett and He (2008), the monetary policy rule is

$$\frac{\dot{i}}{i}=a_p\log\left(\frac{P}{\overline{\overline{P}}}\right)+a_{\text{int}}\frac{\dot{P}}{P}+a_i\log\left(\frac{i}{\overline{\overline{\beta}}}\right)+a_L\log\left(\frac{L}{\overline{\overline{L}}}\right)+\varepsilon_i, \tag{1.3.15}$$

and the tax policy is

$$\frac{d}{dt}\frac{\tau}{C}=b_\tau\left(\frac{\tau}{C}-\frac{\overline{\tau}}{\overline{\overline{C}}}\right)+b_L\log\left(\frac{L}{\overline{\overline{L}}}\right)+b_{\text{inf}}\frac{\dot{P}}{P}+b_x\left(\frac{D}{PY}-\frac{\overline{D}}{\overline{\overline{PY}}}\right)+\varepsilon_\tau\ . \tag{1.3.16}$$

The free parameters are the steady state debt-to-income level, $\bar{D}/\bar{Y}$, the steady state price level, $\bar{P}$, the a's, and the b's. The disturbance noises are ε_i and ε_τ. The control variables are i and τ_c. Barnett and He (2008) use $\tau_C=\frac{\tau}{C}$ rather than τ as a control. The exogenous variables are $n, g, \pi, \delta, \theta, \alpha, A$, and ϕ, which are specified by Leeper and Sims to follow logarithmic first-order autoregressive (AR) processes in continuous time, while β is specified to be a logarithmic first-order AR in unlogged form. Barnett and He (2008) analyze the structural properties of (1.3.1)-(1.3.14) without external disturbances. Barnett and He (2006b, 2008) treat all parameters in (1.3.3) as fixed parameters and treat the exogenous variables as realized at their measured values. The extension of this analysis to the case of stochastic bifurcation is a subject for future research.

Next Barnett and He (2008) reduce the dimension of the problem by temporarily eliminating some state variables for the convenience of analytical investigation. They contract to the following 7 state variables

$$\mathbf{x} = \begin{bmatrix} D \\ P \\ C \\ L \\ K \\ Z \\ Y \end{bmatrix}. \tag{1.3.17}$$

The remaining state variables can be written as unique functions of **x.** By eliminating M, C^*, V, Q, X from the independent state variables, it can be determined directly from (1.3.1)-(1.3.14) that $\mathbf{x}$ satisfies the following equations.

$$\frac{1}{P}\dot{D} + \frac{Y\sqrt{\phi / i}}{P}\dot{P} + \left(\sqrt{\phi / i}\right)\dot{Y}$$

$$= Y + \frac{iD}{P} - \left(\frac{Y}{C+g}\right)^{1-\mu} C - \theta\left(\frac{Y}{Z+nK}\right)^{1-\mu} L - \tau_C C + \frac{Y\sqrt{i/\phi}}{2V^2\phi}\dot{i} \quad , \tag{1.3.18}$$

$$\left(1-\pi(1-\gamma)\right)\left(\frac{1-\phi VY^{\mu}(1-\mu)(C+g)^{-\mu}}{C-\phi VY^{\mu}(C+g)^{1-\mu}}-\frac{1-\mu}{C+g}\right)\dot{C}$$
$$-\left(\frac{(1-\pi(1-\gamma))\phi V\mu Y^{\mu-1}(C+g)^{1-\mu}}{C-\phi VY^{\mu}(C+g)^{1-\mu}}+\frac{1-\mu}{Y}\right)\dot{Y}+\frac{\dot{P}}{P}+\frac{(1-\gamma)(1-\pi)}{1-L}\dot{L} \tag{1.3.19}$$
$$=i-\beta+\frac{Y^{\mu}\left(C+g\right)^{1-\mu}}{C-\phi VY^{\mu}(C+g)^{1-\mu}}\frac{1}{2\sqrt{i\phi}}\dot{i},$$

$$\frac{\dot{P}}{P}+(1-\mu)\left(\frac{\dot{Y}}{Y}-\frac{\dot{Z}+n\dot{K}}{Z+nK}\right)$$

$$=-(1-2\phi V)\frac{a^{\sigma}\alpha}{\theta}Y^{\mu-\theta}\left(Z+nK\right)^{1-\mu}K^{\sigma-1}+i+\delta, \tag{1.3.20}$$

$$\dot{K}=Z-\delta K, \tag{1.3.21}$$

$$0=(C+g)^{\mu}+\theta(Z+nK)^{\mu}-Y^{\mu}, \tag{1.3.22}$$

$$0=\alpha K^{\sigma}+L^{\sigma}-a^{-\sigma}Y^{-\sigma} \tag{1.3.23}$$

$$0=(1-2\phi V)\frac{a^{\sigma}Y^{\mu-\sigma}(C+g)^{1-\mu}}{L^{1-\sigma}}+\frac{1-\pi}{\pi}\frac{\phi V}{1-L}Y^{\mu}(C+g)^{1-\mu}-\frac{1-\pi}{\pi}\frac{C}{1-L}\;. \tag{1.3.24}$$

Then Barnett and He (2008) write equations (1.3.18)--(1.3.24) as

$$\mathbf{h}(\mathbf{x},\mathbf{u})\dot{\mathbf{x}}=\mathbf{f}(\mathbf{x},\mathbf{u}), \tag{1.3.25}$$

$$\mathbf{0}=\mathbf{g}(\mathbf{x},\mathbf{u}), \tag{1.3.26}$$

where $\mathbf{x}$ is a 7-dimensional state vector, $\mathbf{u}$ is a 2-dimensinal control vector, $\mathbf{h}(\mathbf{x},\mathbf{u})$ is a 4×7-dimensional matrix, and $\mathbf{f}(\mathbf{x},\mathbf{u})$ is a 4×1 vector of functions, $\mathbf{g}(\mathbf{x},\mathbf{u})$ is a 3×1 vector of functions. Equation (1.3.25) describes the nonlinear dynamical behavior of the model, and

(1.3.26) describes the nonlinear algebraic constraints. The system formed by (1.3.25) and (1.3.26) is called nonlinear descriptor systems in the mathematical literature. Barnett and He (2006b,2008) use $m = 7, m_1 = 4, m_2 = 3,$ and $l = 2$ (with $m = m_1 + m_2$) to denote respectively the dimension of $\mathbf{x}$, the number of differential equations in (1.3.25), the number of algebraic constraints in (1.3.26), and the dimension of the vector of control variables $\mathbf{u}$.

Barnett and He (2008) solve the steady state of the system (1.3.25)-(1.3.26) for the 7 state variables, $\mathbf{x}$, conditionally on the setting of the controls $\mathbf{u}$ from the following equations:

$$\mathbf{0} = \mathbf{f}(\mathbf{x}, \mathbf{u}), \tag{1.3.27}$$

$$\mathbf{0} = \mathbf{g}(\mathbf{x}, \mathbf{u}). \tag{1.3.28}$$

and get

$$\begin{aligned} \bar{i} &= \beta \\ \bar{\dot{i}} &= 0 \\ \bar{\tau}_c &= \frac{\bar{\tau}}{\bar{C}} \end{aligned} \tag{1.3.29}$$

The first equation of (1.3.29) is found from (1.3.15) in the steady state, the second equation from the definition of steady state, and the third equation from (1.3.16) in the steady state. The values $\bar{\mathbf{x}}$ and $\bar{\mathbf{u}}$ are solutions to (1.3.27)-(1.3.28), and (1.3.29). The resulting steady state is the equilibrium of (1.3.25)-(1.3.26), when the control variables are set at their steady state.

The vector of parameters in the steady state system is

$$\mathbf{p} = [\pi \ \beta \ \theta \ \alpha \ a \ \phi \ \delta \ \mu \ \gamma \ \sigma]'.$$

Here g is taken as a fixed value by the private sector at its setting by the government. The constraints on the parameter values and g are:

$$0 < \pi < 1, \gamma > 0, 0 \le \sigma \le 1, \mu \ge 1, \delta \ge 0, 0 \le \beta \le 1, \delta > 0, g \ge 0. \quad (1.3.30)$$

1.3.3 Singularity in Leeper and Sims Model

Barnett and He (2008) use local linearization around the equilibrium $(\bar{\mathbf{x}}, \bar{\mathbf{u}})$ and derive the following linearized system of (1.3.25) and (1.3.26):

$$\mathbf{E_1}\dot{\mathbf{x}} = \mathbf{A_1}\mathbf{x} + \mathbf{B_1}\mathbf{u}, \quad (1.3.31)$$

$$\mathbf{0} = \mathbf{A_2}\mathbf{x} + \mathbf{B_2}\mathbf{u}, \quad (1.3.32)$$

where

$$\mathbf{E}_1 = \mathbf{h}(\bar{\mathbf{x}}, \bar{\mathbf{u}}) \in R^{m_1 \times m} = R^{4\times 7}$$

$$\mathbf{A}_1 = \frac{\partial \mathbf{f}(\mathbf{x},\mathbf{u})}{\partial \mathbf{x}} \Big|_{\mathbf{x}=\bar{\mathbf{x}}, \mathbf{u}=\bar{\mathbf{u}}} \in R^{m_1 \times m} = R^{4\times 7}$$

$$\mathbf{A}_2 = \frac{\partial \mathbf{g}(\mathbf{x},\mathbf{u})}{\partial \mathbf{x}} \Big|_{\mathbf{x}=\bar{\mathbf{x}}, \mathbf{u}=\bar{\mathbf{u}}} \in R^{m_2 \times m} = R^{3\times 7}$$

$$\mathbf{B}_1 = \frac{\partial \mathbf{f}(\mathbf{x},\mathbf{u})}{\partial \mathbf{u}} \Big|_{\mathbf{x}=\bar{\mathbf{x}}, \mathbf{u}=\bar{\mathbf{u}}} \in R^{m_1 \times l} = R^{4\times 2}$$

$$\mathbf{B}_2 = \frac{\partial \mathbf{g}(\mathbf{x},\mathbf{u})}{\partial \mathbf{u}} \Big|_{\mathbf{x}=\bar{\mathbf{x}}, \mathbf{u}=\bar{\mathbf{u}}} \in R^{m_2 \times l} = R^{3\times 2}$$

Barnett and He (2008) find the linearized system satisfies the regularity condition according to Gantmacher (1974). In particular, they find values of the determinant's parameter s such that $det(\begin{bmatrix} s\mathbf{E_1} - \mathbf{A_1} \\ -\mathbf{A_2} \end{bmatrix}) \not\equiv 0$. Since the linearized system is regular, it is solvable. Barnett and He (2008) further transform the linearized system (1.3.31)-(1.3.32) into the following form.

Definition 1.3.1 (Barnett and He (2008), Definition 3.1) Two systems

$$\mathbf{E\dot{x}} = \mathbf{Ax} + \mathbf{Bu} \tag{1.3.33}$$

and

$$\mathbf{\tilde{E}\dot{y}} = \mathbf{\tilde{A}y} + \mathbf{\tilde{B}u} \tag{1.3.34}$$

are said to be restricted system equivalent (r.s.e), if there exist two nonsingular matrices $\mathbf{T_1}$ and $\mathbf{T_2}$ such that

$$\mathbf{T_1ET_2} = \mathbf{\tilde{E}},\ \ \mathbf{T_1AT_2} = \mathbf{\tilde{A}},\ \ \mathbf{T_1B} = \mathbf{\tilde{B}},\ \ \mathbf{x} = \mathbf{T_2y}.$$

Barnett and He (2008) note that the form (1.3.34) can be obtained by using the coordinate transform $\mathbf{x} = \mathbf{T_2y}$ into (1.3.33) and then multiplying both sides of (1.3.33) by $\mathbf{T_1}$ from the left. They next transformed (1.3.31)-(1.3.32) into suitable r.s.e. forms. They denote $r_E = rank\ (\mathbf{E_1})$, where $r_E \in \{1,2,3,4\}$. Then there exist nonsingular matrices $\mathbf{T_1}$ and $\mathbf{T_2}$ such that

$$\mathbf{T_1E_1T_2} = \begin{bmatrix} \mathbf{I_{r_E}} & 0 \\ 0 & 0 \end{bmatrix}.$$

They substitute the form $\mathbf{x} = \mathbf{T_2}\begin{bmatrix} \mathbf{y_1} \\ \mathbf{y_2} \end{bmatrix}$, where $\mathbf{y_1} \in R^{r_E}$ and $\mathbf{y_2} \in R^{m-r_E} = R^{7-r_E}$, into (1.3.31)-(1.3.32) and also multiply both sides of (1.3.31) by $\mathbf{T_1}$. It follows that (1.3.31)-(1.3.32) is r.s.e to

$$\dot{\mathbf{y}}_1 = \mathbf{A}_{11}\mathbf{y}_1 + \mathbf{A}_{12}\mathbf{y}_2 + \mathbf{B}_{11}\mathbf{u}, \tag{1.3.35a}$$

$$\mathbf{0} = \mathbf{A}_{21}\mathbf{y}_1 + \mathbf{A}_{22}\mathbf{y}_2 + \mathbf{B}_{12}\mathbf{u}, \tag{1.3.35b}$$

$$\mathbf{0} = \mathbf{A}_{31}\mathbf{y}_1 + \mathbf{A}_{32}\mathbf{y}_2 + \mathbf{B}_{2}\mathbf{u}, \tag{1.3.35c}$$

where

$$\begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{bmatrix} = \mathbf{T}_1\mathbf{A}_1\mathbf{T}_2, \quad \begin{bmatrix} \mathbf{B}_{11} \\ \mathbf{B}_{12} \end{bmatrix} = \mathbf{T}_1\mathbf{B}_1, \; [\mathbf{A}_{31} \;\; \mathbf{A}_{32}] = \mathbf{A}_2\mathbf{T}_2\,,$$

with $\mathbf{A}_{11} \in R^{r_E\times r_E}, \mathbf{A}_{12} \in R^{r_E\times(7-r_E)}, \mathbf{A}_{21} \in R^{(4-r_E)\times r_E}, \mathbf{A}_{22} \in R^{(4-r_E)\times(7-r_E)}, \mathbf{A}_{31} \in R^{3\times r_E}, \mathbf{A}_{32} \in R^{3\times(7-r_E)}, \mathbf{B}_{11} \in R^{r_E\times 2}$, and $\mathbf{B}_{12} \in R^{(4-r_E)\times 2}$, while $\mathbf{y}_1$ is an r_E dimensional vector and $\mathbf{y}_2$ is a $7 - r_E$ dimensional vector.

Barnett and He (2008) combine equations (1.3.35a) and (1.3.35b) and acquire the following:

$$\dot{\mathbf{y}}_1 = \mathbf{A}_{11}\mathbf{y}_1 + \mathbf{A}_{12}\mathbf{y}_2 + \mathbf{B}_{11}\mathbf{u}, \tag{1.3.36a}$$

$$\mathbf{0} = \widetilde{\mathbf{A}}_{21}\mathbf{y}_1 + \widetilde{\mathbf{A}}_{22}\mathbf{y}_2 + \widetilde{\mathbf{B}}_{12}\mathbf{u}, \tag{1.3.36b}$$

where

$$\widetilde{\mathbf{A}}_{21} = \begin{bmatrix} \mathbf{A}_{21} \\ \mathbf{A}_{31} \end{bmatrix}, \qquad \widetilde{\mathbf{A}}_{22} = \begin{bmatrix} \mathbf{A}_{22} \\ \mathbf{A}_{32} \end{bmatrix}, \qquad \widetilde{\mathbf{B}}_{12} = \begin{bmatrix} \mathbf{B}_{12} \\ \mathbf{B}_{2} \end{bmatrix}.$$

If $\widetilde{\mathbf{A}}_{22}$ is nonsingular, it follows from (1.3.36b) that $\mathbf{y}_2 = -\left(\widetilde{\mathbf{A}}_{22}\right)^{-1}\left(\widetilde{\mathbf{A}}_{21}\mathbf{y}_1 + \widetilde{\mathbf{B}}_{12}\mathbf{u}\right)$. They substitute the form of $\mathbf{y}_2$ into (1.3.36a) and get

$$\dot{\mathbf{y}}_1 = \mathbf{C}\mathbf{y}_1 + \mathbf{D}\mathbf{u}, \tag{1.3.37}$$

where $\mathbf{C} = \mathbf{A_{11}} - \mathbf{A_{12}}\widetilde{\mathbf{A}}_{22}^{-1}\widetilde{\mathbf{A}}_{21} \in R^{r_E \times r_E}$ and $\mathbf{D} = \mathbf{B_{11}} - \mathbf{A_{12}}\widetilde{\mathbf{A}}_{22}^{-1}\widetilde{\mathbf{B}}_{12} \in R^{r_E \times 2}$. This implies that if $\widetilde{\mathbf{A}}_{22}$ is nonsingular, given the algebraic relationship between $\mathbf{y_1}$ and $\mathbf{y_2}$ in equation (1.3.36b), the dynamics of $\mathbf{y_1}$ can be explained in terms of ordinary differential equations (1.3.37).

Linear system ((1.3.31), (1.3.32)) is equivalent to ((1.3.37), (1.3.36b)), only when $\widetilde{\mathbf{A}}_{22}$ is nonsingular. If $\widetilde{\mathbf{A}}_{22}$ were singular, the above transformation would not be possible and singular bifurcation would occur. As explained in Barnett and He (2004,2006b), if $\widetilde{\mathbf{A}}_{22}$ becomes exactly singular ,the dimension of dynamics change. The dynamics also would change substantially, if $\widetilde{\mathbf{A}}_{22}$ moves between two settings located on opposite sides of a singular bifurcation boundary.

To examine the case when $\widetilde{\mathbf{A}}_{22}$ is singular in more detail, Barnett and He (2008) rewrite the linearized system ((1.3.36a), (1.3.36b)) as

$$\begin{bmatrix} \mathbf{I_{r_E}} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}\begin{bmatrix} \dot{\mathbf{y}}_1 \\ \dot{\mathbf{y}}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{A_{11}} & \mathbf{A_{12}} \\ \widetilde{\mathbf{A}}_{21} & \widetilde{\mathbf{A}}_{22} \end{bmatrix}\begin{bmatrix} \mathbf{y_1} \\ \mathbf{y_2} \end{bmatrix} + \begin{bmatrix} \mathbf{B_{11}} \\ \widetilde{\mathbf{B}}_{12} \end{bmatrix}\mathbf{u}. \tag{1.3.38}$$

The matrix pair $\left(\begin{bmatrix} \mathbf{I_{r_E}} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}, \begin{bmatrix} \mathbf{A_{11}} & \mathbf{A_{12}} \\ \widetilde{\mathbf{A}}_{21} & \widetilde{\mathbf{A}}_{22} \end{bmatrix}\right)$, which is in the form of a matrix pencil, is also regular, since the model is regular. Therefore, there exist nonsingular matrices, $\widetilde{\mathbf{T}}_1$ and $\widetilde{\mathbf{T}}_2$, such that (Gantmacher (1974)):

$$\widetilde{\mathbf{T}}_1\begin{bmatrix} \mathbf{I_{r_E}} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}\widetilde{\mathbf{T}}_2 = \begin{bmatrix} \mathbf{I}_{\widetilde{\mathbf{m}}_1} & \mathbf{0} \\ \mathbf{0} & \mathbf{N} \end{bmatrix} \text{ and } \widetilde{\mathbf{T}}_1\begin{bmatrix} \mathbf{A_{11}} & \mathbf{A_{12}} \\ \widetilde{\mathbf{A}}_{21} & \widetilde{\mathbf{A}}_{22} \end{bmatrix}\widetilde{\mathbf{T}}_2 = \begin{bmatrix} \widetilde{\mathbf{A}}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_{\widetilde{\mathbf{m}}_2} \end{bmatrix},$$

where $\tilde{\mathrm{m}}_1 + \tilde{\mathrm{m}}_2 = \mathrm{m}$ and $\mathbf{N}$ is a nilpotent matrix; i.e. there exists a positive integer $d \geq 1$ such that $\mathbf{N}^d$=0. The smallest such integer d is called the nilpotent index of $\mathbf{N}$. One example of a nilpotent matrix is:

$$\mathbf{N} = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ & \dots & & & \dots . & \\ 0 & 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & 0 & \dots & 0 & 0 \end{bmatrix}. \tag{1.3.39}$$

Barnett and He (2008) next consider the coordinate transform $\begin{bmatrix} \mathbf{y_1} \\ \mathbf{y_2} \end{bmatrix} = \widetilde{\mathbf{T}}_2 \begin{bmatrix} \mathbf{z_1} \\ \mathbf{z_2} \end{bmatrix}$, substitute it for $\mathbf{y}$ in equation (1.3.38), and multiply both sides of (1.3.38) by $\widetilde{\mathbf{T}}_\mathbf{1}$ from the left. The following r.s.e. form of ((1.3.31),(1.3.32)) results:

$$\dot{\mathbf{z}}_\mathbf{1} = \widetilde{\mathbf{A}}_\mathbf{1}\mathbf{z_1} + \widetilde{\mathbf{B}}_\mathbf{1}\mathbf{u}, \tag{1.3.40}$$

$$\mathbf{N}\dot{\mathbf{z}}_\mathbf{2} = \mathbf{z_2} + \widetilde{\mathbf{B}}_\mathbf{2}\mathbf{u}, \tag{1.3.41}$$

where

$$\begin{bmatrix} \widetilde{\mathbf{B}}_\mathbf{1} \\ \widetilde{\mathbf{B}}_\mathbf{2} \end{bmatrix} = \widetilde{\mathbf{T}}_1 \begin{bmatrix} \mathbf{B_{11}} \\ \widetilde{\mathbf{B}}_\mathbf{12} \end{bmatrix}.$$

The solutions to (1.3.40) and (1.3.41) are respectively

$$\mathbf{z_1} = e^{\widetilde{\mathbf{A}}_\mathbf{1}(t-t_0)}\mathbf{z_1}(0) + \int_{t_0}^{t} e^{\widetilde{\mathbf{A}}_\mathbf{1}(t-\xi)}\,\widetilde{\mathbf{B}}_\mathbf{1}\mathbf{u}(\xi)d\xi\ ,$$

$$\mathbf{z_2} = -\sum_{k=1}^{d-1}\delta^{(k-1)}(t)\mathbf{N}^k\mathbf{z_2}(0) - \sum_{k=0}^{d-1}\mathbf{N}^k\widetilde{\mathbf{B}}_\mathbf{2}\mathbf{u}^{(k)}(t),$$

where $t_0 \geq 0$ is the initial time, $\delta^{(k-1)}(t)$ is the derivative of order $k-1$ of the Dirac delta function, and $\mathbf{u}^{(k)}$ denotes that k-th order derivative of $\mathbf{u}$.

If $\mathbf{N} = \mathbf{0}$, it follows from (1.3.41) that $\mathbf{z_2} = -\widetilde{\mathbf{B}}_2\mathbf{u}$, which is a smooth algebraic relationship between $\mathbf{z_2}$ and $\mathbf{u}$; and the above solution for $\mathbf{z_2}$ does not apply. Only when **N** is nonzero, there exist impulsive terms involving the Dirac delta functions, which could produce shock effects in the first summation of the solution for $\mathbf{z_2}$, and smooth derivative terms of $\mathbf{u}$ in the second summation. The solution structure with nonzero **N** is very different from the solution of ordinary differential equations as in (1.3.40) for $\mathbf{z_1}$.

The following theorem links bifurcation phenomenoa at $\mathbf{N} \neq \mathbf{0}$ to the singularity of $\widetilde{\mathbf{A}}_{22}$. The proof is contained in Barnett and He (2008), Theorem 3.1.

Theorem 1.3.1. If both (1.3.40)-(1.3.41) and (1.3.36a)-(1.3.36b) are r.s.e forms of the same linearized system (1.3.31)-(1.3.32), then $\mathbf{N} = \mathbf{0}$, if and only if $\widetilde{\mathbf{A}}_{22}$ is nonsingular. Hence it follows that

$$det\ (\widetilde{\mathbf{A}}_{22}\ ) \neq 0.$$

The next theorem links the singularity of $\widetilde{\mathbf{A}}_{22}$ to the rank of the original coefficient matrix. The proof is contained in Barnett and He (2008), Theorem 1.3.2.

Theorem 1.3.2. Assume that $\mathbf{E_1}$ has full row rank, i.e.

$$rank\ (\mathbf{E_1}) = m_1.$$

Then $\widetilde{\mathbf{A}}_{22}$ is nonsingular, if and only if the $m\times m$ matrix $\begin{bmatrix} \mathbf{E_1} \\ \mathbf{A_2} \end{bmatrix}$ is nonsingular, so that

$$rank\left(\begin{bmatrix} \mathbf{E_1} \\ \mathbf{A_2} \end{bmatrix}\right) = m.$$

Theorem 1.3.2 provides the condition for the existence of a singularity bifurcation boundary, so that $det\left(\begin{bmatrix}\mathbf{E_1}\\ \mathbf{A_2}\end{bmatrix}\right) = 0$.

The following corollary says that the singularity condition does not change whenever state variables that can be modeled by ordinary differential equations are added or deleted. The proof is contained in Barnett and He (2008), Corollary 3.1.

Corollary 1.3.1. Consider the following system describing the dynamics of $(\mathbf{x}, \mathbf{v})$, where $\mathbf{v} \in R^{m_3}$ for arbitrary m_3.

$$\mathbf{E_1\dot{x}} + \mathbf{E_{1v}\dot{v}} = \mathbf{A_1x} + \mathbf{A_{1v}v} + \mathbf{B_1u}, \tag{1.3.42a}$$

$$\mathbf{\dot{v}} = \mathbf{A_v v} + \mathbf{B_v u}, \tag{1.3.42b}$$

$$\mathbf{0} = \mathbf{A_2x} + \mathbf{A_{2v}v} + \mathbf{B_2u}, \tag{1.3.42c}$$

where $\mathbf{E_{1v}}, \mathbf{A_{1v}}, \mathbf{A_v}, \mathbf{B_v}, \mathbf{A_{2v}}$ are arbitrary matrices of dimension

$m_1{\times}m_3, m_1{\times}m_3,\ m_3{\times}m_3, m_3{\times}l,$ and $m_2{\times}m_3$,

respectively, and the other matrices are as defined above. Then the singularity condition for (1.3.42a), (1.3.42b), and (1.3.42c) is the same as that for ((1.3.31), (1.3.32)).

The above corollary says that adding (or deleting) state variable that can be modeled by ordinary differential equations does not change the singularity condition. The corollary is useful in reducing the dimension of the problem under consideration. With this corollary, Barnett and He (2008) are able to drop the Leeper and Sims' model's state variable K from the state vector (1.3.17) in the system ((1.3.31), (1.3.32)) without affecting the singularity condition. The singularity condition then becomes

$$det\left(\begin{bmatrix}\mathbf{E}_1'\\ \mathbf{A}_2'\end{bmatrix}\right)=0, \tag{1.3.43}$$

in which

$$\mathbf{E}_1'=\begin{bmatrix} \frac{1}{p} & \frac{Y}{PV} & 0 & 0 & 0 & \frac{1}{V}\\ 0 & \frac{1}{P} & e_{23} & \frac{(1-\gamma)(1-\pi)}{1-L} & 0 & e_{26}\\ 0 & \frac{1}{P} & 0 & 0 & -\frac{1-\mu}{Z+nK} & \frac{1-\mu}{Y}\end{bmatrix}$$

and

$$\mathbf{A}_2'=\begin{bmatrix} 0 & 0 & \mu(C+g)^{\mu-1} & 0 & \theta\mu(Z+nK)^{\mu-1} & \mu Y^{\mu-1}\\ 0 & 0 & a_{23} & a_{24} & 0 & a_{26}\\ 0 & 0 & 0 & \sigma L^{\sigma-1} & 0 & A^{-\sigma}\sigma Y^{\sigma-1}\end{bmatrix}$$

with

$$e_{23}=\frac{1-\pi(1-\gamma)}{C^*}[1-\phi VY^{\mu}(\mu-1)(C+g)^{\mu-2}]-\frac{1-\mu}{C+g},$$

$$e_{26}=\frac{1-\pi(1-\gamma)}{C^*}[-\phi VY^{\mu}\mu(C+g)^{\mu-1}]+\frac{1-\mu}{Y},$$

$$a_{23}=(1-2\phi V)A^{\sigma}Y^{\mu-\sigma}L^{\sigma-1}(1-\mu)(C+g)^{-\mu}-\frac{1-\pi}{\pi}\frac{1}{1-L},$$

$$a_{24}=(1-2\phi V)A^{\sigma}Y^{\mu-\sigma}(\sigma-1)L^{\sigma-2}(C+g)^{1-\mu}-\frac{1-\pi}{\pi}\frac{C}{(1-L)^2},$$

$$a_{26}=(1-2\phi V)A^{\sigma}(\mu-\sigma)Y^{\mu-\sigma-1}L^{\sigma-1}(C+g)^{1-\mu}.$$

The prime denotes the deletion of the state variable K from the vector $\mathbf{x}$ in equation (1.3.17) and deletion of equation (1.3.21), which is the corresponding differential equation for capital K.

Barnett and He (2008) also show by direct calculation that (1.3.43) is equivalent to

$$det\begin{bmatrix} e_{23} & \frac{(1-\gamma)(1-\pi)}{1-L} & \frac{1-\mu}{Z+nK} & e_{26}' \\ \mu(C+g)^{\mu-1} & 0 & \theta\mu(Z+nK)^{\mu-1} & -\mu Y^{\mu-1} \\ a_{23} & a_{24} & 0 & a_{26} \\ 0 & \sigma L^{\sigma-1} & 0 & A^{-\sigma}\sigma Y^{\sigma-1} \end{bmatrix} = 0$$

(1.3.44)

where

$$e_{26}' = \frac{1-\pi(1-\gamma)}{C^*}[-\phi V Y^{\mu}\mu(C+g)^{\mu-1}].$$

Equation (1.3.44) determines the singularity-induced bifurcation boundary. According to Barnett and He (2008), this is the first time that this type of bifurcation has been found in a macroeconometric model.

To investigate bifurcation of the closed-loop system under the control of the monetary policy rule and tax policy rule introduced in (1.3.15) and (1.3.16), Barnett and He (2008) augment the state variable to include two more controls as follows:

$$\mathbf{x_c} = \begin{bmatrix} D \\ P \\ C \\ L \\ K \\ Z \\ Y \\ i \\ \tau_c \end{bmatrix}. \qquad (1.3.45)$$

The corresponding linearized system (1.3.31)-(1.3.32) becomes

$$\mathbf{E}_1^c \dot{\mathbf{x}}_c = \mathbf{A}_1^c \mathbf{x}_c, \tag{1.3.46}$$

$$\mathbf{0} = [\mathbf{A}_2 \quad \mathbf{0}] \mathbf{x}_c, \tag{1.3.47}$$

where $\mathbf{E}_1^c \in R^{m_1^c \times m^c} = R^{6\times 9}$, $\mathbf{A}_1^c \in R^{m_1^c \times m^c} = R^{6\times 9}$, $m_1^c = m_1 + 2$, $m^c = m + 2$.

1.3.4 Numerical Results

Corollary 1.3.1 allows adding (or deleting) state variables that can be modeled by ordinary differential equations without changing the singularity condition. Barnett and He (2008) then apply condition (1.3.44) to the closed-loop system (1.3.47) and look for bifurcation boundaries. They vary pairs of parameters with all other parameters set at their estimates. They also find the intersection of their theoretically feasible ranges and the 95% confidence intervals of their estimated values, in particular, the intersection $\boldsymbol{\Xi}$ of (1.3.30) and $[\bar{p}(i) - \bar{c}\sigma_i, \bar{p}(i) + \bar{c}\sigma_i]$, where $\bar{p}(i)$ is the estimated value of parameter $p(i)$, σ_i is the standard error of the estimate, and $\bar{c}$ is the critical value of the 95th-percentile confidence interval for $N(0,1)$.

The estimation information for the parameters μ, g, and β is taken directly from the Leeper and Sims paper, which is presented in Table 1.3.1[8].

[8] Table 1.3.1 is a replicate of Barnett and He (2008), Table 1.

Table 1.3.1 Estimation of μ, g, and β

Parameter	Estimate	Standard Error	Ξ Interval
μ	1.0248	0.324	[1, 1.6598]
g	0.0773	0.292	[0, 0.6496]
β	0.1645	0.288	[0, 0.7290]

Note: Since g is an exogenous variable, rather than a parameter, the "estimate" is the sample mean, and the "standard error" is the sample standard deviation.

Barnett and He (2008) display a few representative sections of the singularity bifurcation boundary. One section is μ versus g, the other is μ versus β. They then explore what happens when β crosses the singularity boundary, with β ranging between 0.08 and 0.24. Table 1.3.2 displays the changes of finite eigenvalues, $\lambda_1, \ldots, \lambda_8$, corresponding to the changes of β.[9]

Table 1.3.2 Eigenvalue Changes

β	0.080	0.120	0.160	0.165	0.170	0.200	0.240
λ_1	1.002	1.002	1.002	1.002	1.002	1.002	1.002
λ_2	0.080	0.120	0.160	0.165	0.170	0.200	0.240
λ_3	-0.303	-0.262	-0.220	-0.215	-0.210	-0.178	-0.135
λ_4	-3.558	-3.559	-3.561	-3.561	-3.561	-3.563	-3.566
λ_5	-0.098	-0.084	-0.077	-0.076	-0.075	-0.072	-0.069
λ_6	-0.002	-0.003	-0.003	-0.003	-0.003	-0.004	-0.004
λ_7	3.101	5.177	8.237	8.682	9.254	13.416	28.401
λ_8	-117.790	-204.703	-1811.413	∞	1456.294	195.888	58.059

Three more infinite eigenvalues are not shown in Table 1.3.2. The second through the ninth rows are the corresponding finite eigenvalues of the linearized model at each setting of

[9] Table 1.3.2 is a replicate of Barnett and He (2008), Table 2.

β shown in the first row. Table 1.3.2 shows that when the value of β crosses the bifurcation boundary, with β ranging between 0.08 and 0.24, λ_8 decreases from negative values rapidly to $-\infty$, jumps suddenly from $-\infty$ to $+\infty$, and then decreases while remaining positive. This phenomenon shows that the model has a change in dynamic structure, when β crosses the singularity-induced bifurcation boundary. The two regions separated by the boundary exhibit drastically different dynamical behaviors. Barnett and He (2008) also display that very small changes in μ can cause bifurcation independently of the setting of g or β. They also state that the number of dynamic equations and the number of algebraic equations change, when the singularity-induced bifurcation boundary is reached.

1.4 New Keynesian Model[10]

1.4.1. Introduction

This section surveys Barnett and Duzhak's (2008, 2010) work on bifurcation analysis within the class of New Keynesian models. Their interest in exploring bifurcation in New Keynesian models is driven by the increasing policy interest in New Keynesian models. In Barnett and Duzhak (2008, 2010), they have studied different specifications of monetary policy rules within the New Keynesian functional structure and have found both the existence of Hopf bifurcation and the existence of period doubling (flip) bifurcation boundaries through numerical procedures.

The usual New Keynesian log-linearized model consists of a forward-looking IS-curve describing consumption smoothing behavior, a Phillips curve derived from price

[10] This section is summarized from Barnett and Duzhak (2008,2010).

optimization by monopolistically competitive firms in the presence of nominal rigidities, and a monetary policy rule having different specifications. Barnett and Duzhak (2010) use eigenvalues of the linearized system to locate Hopf bifurcation boundaries and investigate different monetary policy effects on bifurcation boundary locations for each case. They use two types of New Keynesian models: one can be reduced to produce a 2×2 Jacobian, and the other produces a 3×3 Jacobian. In the 3×3 case, Barnett and Duzhak (2010) employ a theorem on Hopf bifurcation from the engineering literature.

Starting from Grandmont's findings with a classical model, Barnett and Duzhak (2008, 2010) continue to follow the path from the Bergstrom-Wymer UK model, then to the Euler equations Leeper and Sims' macroeconometric models, and then to New Keynesian models. Barnett and Duzhak (2008, 2010) believe that Grandmont's conclusions appear to hold for all categories of dynamic macroeconomic models and suggest that Barnett and He's initial findings with the Bergstrom-Wymer 's UK model appear to be generic. Barnett and Duzhak (2008, 2010) suggest that study of the full nonlinear system and analysis of continuous-time New Keynesian models will merit future research.

1.4.2. The Model[11]

The main assumption of New Keynesian economic theory is that there are nominal price rigidities preventing prices from adjusting immediately and thereby creating disequilibrium unemployment. Price stickiness is often introduced in the manner proposed by Calvo (1983). The model used by Barnett and Duzhak (2008, 2010) is based upon Walsh

[11] The model description is modified from Barnett and Duzhak (2010).

(2003), section 5.4.1, pp. 232-239, which in turn is based upon the monopolistic competition model of Dixit and Stiglitz (1977).

The model consists of consumers, firms, and monetary policy authority. The representative consumer can allocate wealth to money and bonds and choose the aggregate consumption stream to maximize the utility. Consumers derive utility from the composite consumption good C_t, real money balances, and leisure, and supply labor in a competitive labor market, while receiving labor income $w_t N_t$. Consumers own the firms, which produce consumption goods, and they receive all profits π_t.

Firms operate in a monopolistically competitive market, in which each firm has pricing power over the goods it sells. A random fraction of firms does not adjust its product price in each period. A result is price rigidity by the firm, while the remaining firms adjust prices to their optimal levels. Firms make their production and price-setting decisions by solving the cost minimization and pricing decision problems, such that

$$x_t = E_t x_{t+1} - \frac{i_t - E_t \pi_{t+1}}{\sigma} \tag{1.4.1}$$

$$\pi_t = \beta E_t \pi_{t+1} + \kappa x_t \tag{1.4.2}$$

where π_t is the inflation rate at time t; i_t is the interest rate; $x_t = (\hat{y}_t - \hat{y}_t^f)$ is the gap between actual output percentage deviation $\hat{y}_t$ and the flexible-price output percentage deviation $\hat{y}_t^f$; σ is a degree of relative risk aversion; E_t is the expectations operator, conditionally upon information at time t, and β is the discount factor.

Equation (1.4.1) represents the demand side of the economy and is a forward-looking IS curve that relates the output gap to the real interest rate. Equation (1.4.2) represents the supply side and is the New-Keynesian Phillips curve describing how inflation is driven by the output gap and expected inflation. The remaining equation to close the model will be a monetary policy rule, in which the central bank uses a nominal interest rate as the policy instrument. Two main policy classes are targeting rules and instrument rules. A well-known instrument rule is Taylor's rule, using a reaction function responding to inflation and output to set the path of the Federal Funds rate. Barnett and Duzhak (2010) initially center analysis on specification of the current-looking Taylor rule, then on forward-looking, backward-looking, and hybrid Taylor rules. Literature also proposes many ways to define an inflation target. Barnett and Duzhak (2010) consider current-looking, forward-looking and backward-looking inflation targeting policies.

1.4.3. Determinacy and Stability Analysis

Barnett and Duzhak (2010) use Theorem 1.1 for the analysis of the reduced 2×2 case of $\mathbf{A}E_t\mathbf{x}_{t+1} = \mathbf{B}\mathbf{x}_t$. They also find bifurcations in the 3×3 case by using the following Lemma 1.4.1 and Theorem 1.4.1, which arise from the engineering literature. That approach had not previously been used in the economics literature. According to Barnett and Duzhak (2010), in the 3×3 case with current-looking or backward-looking policy rules, the only form of bifurcation detected from the linearized model was Hopf bifurcation.

Lemma 1.4.1. (Barnett and Duzhak (2010), Lemma 3.1) For a matrix $\mathbf{A} = [a_{ij}]$, with $i, j = 1,2,3$, a pair of complex conjugate eigenvalues lies on the unit circle and another eigenvalue lies inside the unit circle, if and only if

(a) $|x| < 1$,

(b) $|x + z| < 1 + y$,

$(c)\quad y - xz = 1 - x^2,$

where $z, y,$ and x are the coefficients of the characteristic equation $\lambda^3 + z\lambda^2 + y\lambda + x = 0$ of the matrix **A.**

The following theorem is Barnett and Duzhak's (2010), Theorem 3.2. The proof is included in that paper.

Theorem 1.4.1 (Existence of Hopf Bifurcation in 3 Dimensions) Consider a map $\mathbf{x} \mapsto \mathbf{f}(\mathbf{x}, \boldsymbol{\varphi})$, where $\mathbf{x}$ has 3 dimensions. Let $\mathbf{J}$ be the Jacobian of the transformation, and let the characteristic polynomial of the Jacobian be $P(\lambda) = \lambda^3 + z\lambda^2 + y\lambda + x = 0$. Assume that for one of the equilibria, $(\mathbf{x}^*, \boldsymbol{\varphi}^*)$, there is a critical value, φ_i^c, for one of the parameters, φ_i^*, in $\boldsymbol{\varphi}^*$such that eigenvalue conditions (a),(b), and (c) and transversality condition (d) hold, where:

$(a)\quad |x| < 1,$

$(b)\quad |x + z| < 1 + y,$

$(c)\quad y - xz = 1 - x^2,$

$(d)\quad \dfrac{\partial \left|\lambda_j(\mathbf{x}^*, \boldsymbol{\varphi}^*)\right|}{\partial \varphi_i^*}\Big|_{\varphi_i^* = \varphi_i^c} \neq 0$ for the complex conjugates with $j = 1,2.$

Then there is an invariant closed curve Hopf-bifurcating from $\boldsymbol{\varphi}^*$.

i. Current-Looking Taylor Rule

The current-looking Taylor rule is:

$$i_t = a_1\pi_t + a_2 x_t, \tag{1.4.3}$$

where a_1 is the coefficient of the central bank's reaction to inflation and a_2 is the coefficient of the central bank's reaction to the output gap.

The 3-equation system ((1.4.1), (1.4.2), (1.4.3)) constitutes a New Keynesian model. To analyze the model's determinacy and stability properties, Barnett and Duzhak (2010) first display the system in the following form, which is not a closed form:

$$\mathbf{A}E_t\mathbf{x}_{t+1} = \mathbf{B}\mathbf{x}_t + \boldsymbol{\delta}_t,$$

where

$$\mathbf{A} = \begin{bmatrix} 1 & \frac{1}{\sigma} & 0 \\ 0 & \beta & 0 \\ 0 & 0 & 0 \end{bmatrix},\ \mathbf{B} = \begin{bmatrix} 1 & 0 & \frac{1}{\sigma} \\ -\kappa & 1 & 0 \\ a_2 & a_1 & -1 \end{bmatrix},\ \mathbf{x}_t = \begin{bmatrix} x_t \\ \pi_t \\ i_t \end{bmatrix}.$$

Obtaining the matrix $\mathbf{C} = \mathbf{A}^{-1}\mathbf{B}$ is impossible, since A is a singular matrix.

Therefore, they reduce the system to a system of two log-linearized equations by substituting Taylor's rule (1.4.3) into the consumption Euler equation. The system of two equations has the following form:

$$\begin{bmatrix} 1 & \frac{1}{\sigma} \\ 0 & \beta \end{bmatrix}\begin{bmatrix} E_t x_{t+1} \\ E_t \pi_{t+1} \end{bmatrix} = \begin{bmatrix} 1 + \frac{a_2}{\sigma} & -\frac{a_1}{\sigma} \\ -\kappa & 1 \end{bmatrix}\begin{bmatrix} x_t \\ \pi_t \end{bmatrix},$$

which can be written as

$$\mathbf{A}E_t\mathbf{x}_{t+1} = \mathbf{B}\mathbf{x}_t,$$

where

$$\mathbf{x}_t = \begin{bmatrix} x_t \\ \pi_t \end{bmatrix}, \mathbf{A} = \begin{bmatrix} 1 & \frac{1}{\sigma} \\ 0 & \beta \end{bmatrix}, \text{ and } \mathbf{B} = \begin{bmatrix} 1 + \frac{a_2}{\sigma} & -\frac{a_1}{\sigma} \\ -\kappa & 1 \end{bmatrix}.$$

Premultiply the system by the inverse matrix $\mathbf{A^{-1}}$,

$$\mathbf{A^{-1}} = \begin{bmatrix} 1 & -\frac{1}{\beta\sigma} \\ 0 & \frac{1}{\beta} \end{bmatrix},$$

results in

$$E_t \mathbf{x_{t+1}} = \mathbf{C}\mathbf{x_t} \quad \text{or} \begin{bmatrix} E_t x_{t+1} \\ E_t \pi_{t+1} \end{bmatrix} = \begin{bmatrix} 1 + \frac{a_2\beta + \kappa}{\sigma\beta} & \frac{\alpha_1\beta - 1}{\sigma\beta} \\ -\frac{\kappa}{\beta} & \frac{1}{\beta} \end{bmatrix} \begin{bmatrix} x_t \\ \pi_t \end{bmatrix},$$

where $\mathbf{C} = \mathbf{A^{-1}B}$.

The eigenvalues of $\mathbf{C}$ are the roots of the characteristic polynomial

$$p(\lambda) = det(\mathbf{C} - \lambda\mathbf{I}) = \lambda^2 - \lambda\left[1 + \frac{a_2\beta + \kappa}{\sigma\beta} + \frac{1}{\beta}\right] + \frac{\sigma\beta + a_2\beta + \kappa a_1\beta}{\sigma\beta^2}.$$

Define D as

$$D = \left[1 + \frac{a_2\beta + \kappa}{\sigma\beta} + \frac{1}{\beta}\right]^2 - 4\frac{\sigma\beta + a_2\beta + \kappa a_1\beta}{\sigma\beta^2}.$$

Then the eigenvalues are

$$\lambda_1 = \frac{1}{2}\left(1 + \frac{a_2\beta + \kappa}{\sigma\beta} + \frac{1}{\beta} + \sqrt{D}\right) \quad \text{and} \quad \lambda_2 = \frac{1}{2}\left(1 + \frac{a_2\beta + \kappa}{\sigma\beta} + \frac{1}{\beta} - \sqrt{D}\right).$$

According to Blanchard and Kahn (1980), the system of expected difference equations has a determinate solution, if the number of eigenvalues outside the unit circle equals the number of forward looking variables. This system has two forward-looking variables, x_{t+1} and π_{t+1}. Therefore the stability and uniqueness of the solution require both eigenvalues to be outside the unit circle. It can be shown that both eigenvalues will be outside the unit circle, if and only if

$$(a_1 - 1)\kappa + (1 - \beta)a_2 > 0. \tag{1.4.4}$$

Interest rate rules that satisfy $a_1 > 1$ are called active. Such active rules define Taylor's principle, stating that the interest rate should be set higher than the increase in inflation. When $a_1 > 1$, clearly (1.4.4) holds. Monetary policy satisfying the Taylor's principle is thought to eliminate equilibrium multiplicities.

In this case, the Jacobian of the New Keynesian model can be written in the form:

$$\mathbf{J} = \begin{bmatrix} 1 + \dfrac{a_2\beta + \kappa}{\sigma\beta} & \dfrac{a_1\beta - 1}{\sigma\beta} \\ -\dfrac{\kappa}{\beta} & \dfrac{1}{\beta} \end{bmatrix}.$$

The model is parameterized by:

$$\boldsymbol{\varphi} = \begin{pmatrix} \beta \\ \sigma \\ \kappa \\ a_1 \\ a_2 \end{pmatrix}$$

Barnett and Duzhak (2008, 2010) use a_1 and a_2 as candidates for bifurcation parameters. They employ Theorem 1.1.1 to look for the existence of Hopf bifurcation for this

New Keynesian model with current looking Taylor rule. The following result is proved in Barnett and Duzhak's (2008), Proposition 1.3.1:

Proposition 1.4.1 The new Keynesian model with current-looking Taylor rule, equations (1.4.1),(1.4.2) and (1.4.3), undergoes a Hopf bifurcation at equilibrium points, if and only if the discriminant of the characteristic equation is negative and $a_2^c = \sigma\beta - \kappa a_1 - \sigma$.

Based on the result in Prop 1.4.1, Barnett and Duzhak (2010) find that the bifurcation boundary is the set of parameter values satisfying the following condition:

$$-1 < \frac{\sigma + \sigma\beta - \kappa a_1 \beta + \kappa}{\sigma\beta^2} < 1.$$

ii. Forward-Looking Taylor Rule

A forward-looking Taylor rule is:

$$i_t = a_1 E_t \pi_{t+1} + a_2 E_t x_{t+1}. \tag{1.4.5}$$

The model consisting of (1.4.1),(1.4.2) and (1.4.5) is parameterized by

$$\boldsymbol{\varphi} = \begin{pmatrix} \beta \\ \sigma \\ \kappa \\ a_1 \\ a_2 \end{pmatrix}.$$

The resulting Jacobian has the following form:

$$\mathbf{J} = \begin{bmatrix} \dfrac{\sigma}{\sigma - a_2} + \dfrac{\kappa(1-a_1)}{(\sigma - a_2)\beta} & \dfrac{a_1 - 1}{(\sigma - a_2)\beta} \\ -\dfrac{\kappa}{\beta} & \dfrac{1}{\beta} \end{bmatrix}.$$

Barnett and Duzhak (2010) use a_1 and a_2 as candidates for bifurcation parameters. The following result is proved in Barnett and Duzhak (2008) as Proposition 3.2:

Proposition 1.4.2. The new Keynesian model with forward-looking Taylor rule, equations (1.4.1), (1.4.2) and (1.4.5), undergoes a Hopf bifurcation at equilibrium points, if and only if the discriminant of the characteristic equation is negative and $a_2^C = -\frac{\sigma}{\beta} + \sigma$.

Based on the result in Prop 1.4.2, Barnett and Duzhak (2010) find the bifurcation boundary is the set of parameter values satisfying the following condition:

$$-1 < \frac{1}{2}\left(\beta + \frac{\kappa(1-a_1)}{\sigma} + \frac{1}{\beta}\right) < 1.$$

Barnett and Duzhak (2010) propose a numerical algorithm to detect a period doubling bifurcation, which is based on the following technique. Given the i^{th} iterate of the fixed point, $f^i(\mathbf{x}) - \mathbf{x} = 0$, a period-doubling bifurcation will occur whenever $\varphi_{PD} = 0$ with $\varphi_{PD} = det(\mathbf{J}^{(i)} + \mathbf{I_n})$, where $\mathbf{J}^{(\mathbf{i})}$ is the Jacobian matrix of the iterated map f^i.

Barnett and Duzhak (2010) use the software continuation package CONTENT, developed by Yuri Kuznetsov and V.V. Levitin, to locate the bifurcation boundary. Barnett and Duzhak select the parameter a_2 to be a free bifurcation parameter and find a period-doubling bifurcation point at $a_2 = 2.994$, with the other parameters set constant in accordance with their paper's appendix table. The nature of the state space solution depends upon where the bifurcation boundary is located. If parameter a_2 is moved to 3 with the other parameters set constant, the solution becomes periodic. Along the bifurcation boundary, the values of parameter, a_2, are between 2.75 and 3. When values of $a_1$and a_2 are along the bifurcation boundary with the forward looking Taylor rule, Barnett and Duzhak (2010) find that the central bank actively reacts to the expected future values of inflation and even more aggressively to the forecasted values of the output gap.

iii. Hybrid Taylor Rule

A Hybrid-Taylor rule is:

$$i_t = a_1 E_t \pi_{t+1} + a_2 x_t \tag{1.4.6}$$

This rule was proposed in Clarida, Gali, and Gertler (2000), who maintain that the rule reflects the Federal Reserve's existing policy.

The system ((1.4.1), (1.4.2), (1.4.6)) has the following Jacobian:

$$\mathbf{J} = \begin{bmatrix} 1 + \frac{a_2}{\sigma} + \frac{\kappa(1-a_1)}{\sigma\beta} & \frac{a_1 - 1}{\sigma\beta} \\ -\frac{\kappa}{\beta} & \frac{1}{\beta} \end{bmatrix}.$$

Barnett and Duzhak (2010) use a_1 and a_2 as candidates for bifurcation parameters. The following result was proved in Barnett and Duzhak (2008), Proposition 3.3:

Proposition 1.4.3. The new Keynesian model with Hybrid-Taylor rule, equations, (1.4.1), (1.4.2), and (1.4.6), undergoes a Hopf bifurcation at equilibrium points, if and only if the discriminant of the characteristic equation is negative and $a_2^c = \beta\sigma - \sigma$.

Based on Proposition 1.4.3, Barnett and Duzhak (2010) find that the bifurcation boundary is the set of parameter values satisfying the following condition:

$$-1 < \frac{\sigma(1+\beta^2) + \kappa(1-a_1)}{2\sigma\beta} < 1.$$

iv. Current-Looking Inflation Targeting

The inflation targeting equation is:

$$i_t = a_1 \pi_t, \tag{1.4.7}$$

which can be used instead of the Taylor rule to complete the New Keynesian model.

The system ((1.4.1), (1.4.2), (1.4.7)) has the following Jacobian:

$$\mathbf{J} = \begin{bmatrix} \dfrac{\sigma\beta + \kappa}{\sigma\beta} & \dfrac{a_1\beta - 1}{\sigma\beta} \\ \dfrac{-\kappa}{\beta} & \dfrac{1}{\beta} \end{bmatrix}$$

The model is characterized by

$$\boldsymbol{\varphi} = \begin{pmatrix} \beta \\ \sigma \\ \kappa \\ a_1 \end{pmatrix}.$$

Barnett and Duzhak (2010) use a_1 as a candidate for a bifurcation parameter. The following result is proved in Barnett and Duzhak (2008), Proposition 3.4:

Proposition 1.4.4. The new Keynesian model with current-looking inflation targeting, equations (1.4.1), (1.4.2) and (1.4.7), produces a Hopf bifurcation at equilibrium points, if and only if the discriminant of the characteristic equation is negative and $a_1^C = \dfrac{\sigma\beta - \sigma}{\kappa}$.

Based on Proposition 1.4.4, Barnett and Duzhak (2010) find that the bifurcation boundary is the set of parameter values satisfying the following condition:

$$-3 < \frac{\sigma + \kappa}{\sigma\beta} < 1.$$

v. Forward-Looking Inflation Targeting

A forward-looking inflation targeting rule is:

$$i_t = a_1 E_t \pi_{t+1}. \tag{1.4.8}$$

The system ((1.4.1), (1.4.2), (1.4.8)) has the Jacobian as follows:

$$\mathbf{J} = \begin{bmatrix} 1 + \frac{\kappa(1-a_1)}{\sigma\beta} & \frac{a_1 - 1}{\sigma\beta} \\ -\frac{\kappa}{\beta} & \frac{1}{\beta} \end{bmatrix}.$$

The model is parameterized by

$$\boldsymbol{\varphi} = \begin{pmatrix} \beta \\ \sigma \\ \kappa \\ a_1 \end{pmatrix}.$$

The following proposition is proved in Barnett and Duzhak (2008), Proposition 3.5:

Proposition 1.4.5. The new Keynesian model with forward-looking inflation targeting, equations (1.4.1), (1.4.2), and (1.4.8), produces a Hopf bifurcation at equilibrium points, if and only if the discriminant of the characteristic equation is negative and $\beta^c = 1$.

Based on Proposition 1.4.5, which does not depend on a_1, Barnett and Duzhak (2010) find that the bifurcation boundary is the set of parameter values satisfying the following condition:

$$-3 < \frac{\kappa(a_1 - 1)}{2\sigma} < 1.$$

Parameter β is both the discount factor and the coefficient in (1.4.2) which scales the impact of expected inflation. Assuming for simplicity that $\beta = 1$, Barnett and Duzhak (2010) find it surprising that this common setting of parameter β can put the model directly onto a Hopf bifurcation boundary. This conclusion is conditional upon the assumption that the model is a good approximation to the economy and that the discriminant of the characteristic equation is negative. In such cases, it is not appropriate to set $\beta = 1$.

Barnett and Duzhak (2010) further find that the dynamic solution in phase space, i.e. with inflation rate plotted against output gap, will be periodic, if $\beta = 0.98$. They find that if the parameter value is located directly on the bifurcation boundary, solution in phase space will become an invariant limit cycle.

vi. Backward-Looking Taylor Rule

Backward-looking monetary policy rules are intended to prevent expectations driven fluctuations. Such rules are constructed with decisions based on observed past values of variables. Examples are found in Carlstrom and Fuerst (2000) and Eusepi (2005). Barnett and Duzhak (2010) observe that such a policy should be sufficient for determinacy of equilibria.

In a backward-looking Taylor rule, the central bank sets an interest rate according to the past values of inflation and output gap as follows:

$$i_t = a_1\pi_{t-1} + a_2 x_{t-1}. \qquad (1.4.9)$$

The system ((1.4.1), (1.4.2), (1.4.9)) can be written in the following form:

$$E_t\mathbf{x}_{t+1} = \mathbf{C}\mathbf{x}_t,$$

with

$$\mathbf{C} = \begin{bmatrix} 1+\frac{\kappa}{\sigma\beta} & -\frac{1}{\sigma\beta} & \frac{1}{\sigma} \\ -\frac{\kappa}{\beta} & \frac{1}{\beta} & 0 \\ a_2 & a_1 & 0 \end{bmatrix}, \quad \mathbf{x}_t = \begin{bmatrix} x_t \\ \pi_t \\ i_t \end{bmatrix}.$$

Matrix **C** has the characteristic polynomial

$$p(\lambda) = \det(\mathbf{C} - \lambda\mathbf{I}) = \lambda^3 - \frac{\sigma(1+\beta)+\kappa}{\sigma\beta}\lambda^2 + \frac{\sigma - \beta a_2}{\sigma\beta}\lambda + \frac{\kappa a_1 + a_2}{\sigma\beta}.$$

The following proposition is proved in Barnett and Duzhak (2010), Proposition 3.6.

Proposition 1.4.6. The New Keynesian model with backward-looking Taylor rule produces a Hopf bifurcation at equilibrium points, if the transversality condition $\frac{\partial|\lambda_j(\mathbf{x}^*,\boldsymbol{\varphi}^*)|}{\partial\varphi_i^*}\Big|_{\varphi_i^*=\varphi_i^c} \neq 0$ holds, and if the parameters α_1 and α_2 satisfy the following three conditions at the equilibrium:

(a) $\left|\frac{a_2+\kappa a_1}{\sigma\beta}\right| < 1,$

(b) $a_2(1-\beta)+\kappa(a_1-1) > 0,$

(c) $\frac{\sigma-\beta a_2}{\sigma\beta} + \frac{(\kappa a_1 + a_2)(\sigma(1+\beta)+\kappa)}{\sigma^2\beta^2} = 1 - (\frac{\kappa a_1 + a_2}{\sigma\beta})^2.$

vii. Backward-Looking Inflation Targeting

A backward-looking inflation targeting rule sets the interest rate according to inflation during a previous period, as follows:

$$i_t = a_1\pi_{t-1}. \qquad (1.4.10)$$

The system ((1.4.1), (1.4.2), (1.4.10)) has the Jacobian as follows:

$$\mathbf{J} = \begin{bmatrix} 1+\dfrac{\kappa}{\sigma\beta} & -\dfrac{1}{\sigma\beta} & \dfrac{1}{\sigma} \\ -\dfrac{\kappa}{\beta} & \dfrac{1}{\beta} & 0 \\ 0 & a_1 & 0 \end{bmatrix}.$$

The Jacobian has the characteristic polynomial

$$p(\lambda) = \lambda^3 - \frac{\sigma(1+\beta)+\kappa}{\sigma\beta}\lambda^2 + \frac{1}{\beta}\lambda + \frac{\kappa a_1}{\sigma\beta}.$$

The following proposition is proved in Barnett and Duzhak (2010) as Proposition 3.7.

Proposition 1.4.7 The New Keynesian model with backward-looking inflation targeting produces a Hopf bifurcation at equilibrium points, if the transversality condition $\frac{\partial|\lambda_j(\mathbf{x}^*,\boldsymbol{\varphi}^*)|}{\partial\varphi_i^*}\Big|_{\varphi_i^*=\varphi_i^c} \neq 0$ holds, and if the parameters φ_i^* satisfy the following three conditions at the equilibrium:

$$(a)\quad \left|\frac{\kappa a_1}{\sigma\beta}\right| < 1,$$

$$(b)\quad a_1 > 1,$$

$$(c)\quad \frac{\sigma^2\beta + \kappa a_1(\sigma(1+\beta)+\kappa)}{\sigma^2\beta^2} = 1 - \left(\frac{\kappa a_1}{\sigma\beta}\right)^2.$$

Barnett and Duzhak (2010) note that their numerical search for bifurcations in this class of models has found only Hopf bifurcations.

viii. Current-Looking Taylor Rule with Interest Rate Smoothing Term

A current-looking Taylor rule with interest rate smoothing term allows central bankers to avoid volatility in interest rate by including a lagged interest rate term in the rule as follows:

$$i_t = (1 - a_3)(a_1\pi_t + a_2 x_t) + a_3 i_{t-1}. \tag{1.4.11}$$

Parameter a_3, which is assumed to be between 0 and 1, describes the degree of interest rate smoothing by the central bank. The model consisting of (1.4.1), (1.4.2) and (1.4.11) is parameterized by

$$\boldsymbol{\varphi} = \begin{pmatrix} \beta \\ \sigma \\ \kappa \\ a_1 \\ a_2 \\ a_3 \end{pmatrix}.$$

The model has the following matrix form:

$$E_t\mathbf{x}_{t+1} = \mathbf{C}\mathbf{x}_t,$$

with

$$\mathbf{C} = \begin{bmatrix} 1 + \dfrac{\kappa}{\sigma\beta} & -\dfrac{1}{\sigma\beta} & \dfrac{1}{\sigma} \\ -\dfrac{\kappa}{\beta} & \dfrac{1}{\beta} & 0 \\ -a_2(a_3 - 1) + \dfrac{(-1+a_3)(a_1\sigma - a_2)\kappa}{\sigma\beta} & -\dfrac{(-1+a_3)(a_1\sigma - a_2)}{\sigma\beta} & -\dfrac{a_2(-1+a_3)}{\sigma} + a_3 \end{bmatrix}$$

and

$$\mathbf{x}_t = \begin{bmatrix} x_t \\ \pi_t \\ i_t \end{bmatrix}.$$

This system has the following characteristic polynomial:

$$p(\lambda) = \lambda^3 + \left(\frac{a_2(a_3-1)}{\sigma} - 1 - a_3 - \frac{\kappa}{\sigma\beta} - \frac{1}{\beta} \right) \lambda^2$$

$$+ \left(\frac{\kappa a_1 - a_2 a_3 + a_2 + \kappa a_3(1-a_1)}{\sigma\beta} + a_3 + \frac{1+a_3}{\beta} \right) \lambda - \frac{a_3}{\beta}.$$

The following proposition is proved in Barnett and Duzhak (2010), Proposition 3.8.

Proposition 1.4.8. The New Keynesian model consisting of ((1.4.1), (1.4.2), (1.4.11)) produces a Hopf bifurcation at equilibrium points, if the transversality condition $\frac{\partial |\lambda_j(\mathbf{x}^*, \boldsymbol{\varphi}^*)|}{\partial \varphi_i^*} \Big|_{\varphi_i^* = \varphi_i^c} \neq 0$ holds, and if the parameters, $\boldsymbol{\varphi}^*$, satisfy the following three conditions at the equilibrium:

(a) $a_3 - \beta < 0$,

(b) $a_1 > 1$,

(c) $\frac{1-a_3^2}{\beta} - (1-a_3) + \frac{a_3(a_3-1)}{\beta^2} + \frac{a_3 a_2(a_3-2) + \kappa a_1(1-a_3) + a_2 + \kappa a_3}{\sigma\beta} = 0.$

ix. Backward-Looking Taylor Rule With Interest Rate Smoothing Term

The backward-looking Taylor rule with interest rate smoothing is:

$$i_t = (1 - a_3)(a_1 \pi_{t-1} + a_2 x_{t-1}) + a_3 i_{t-1}. \tag{1.4.12}$$

The model consisting of (1.4.1), (1.4.2) and (1.4.12) has the following Jacobian:

$$\mathbf{J} = \begin{bmatrix} 1 + \dfrac{\kappa}{\sigma\beta} & -\dfrac{1}{\sigma\beta} & \dfrac{1}{\sigma} \\ -\dfrac{\kappa}{\beta} & \dfrac{1}{\beta} & 0 \\ a_2(1-a_3) & a_1(1-a_3) & a_3 \end{bmatrix}$$

with characteristic polynomial

$$p(\lambda) = \lambda^3 - \left(1 + a_3 + \frac{\kappa}{\sigma\beta} + \frac{1}{\beta}\right)\lambda^2 + \left(\frac{a_2\beta(a_3 - 1) + \kappa a_3 + \sigma(1 + a_3)}{\sigma\beta} + a_3\right)\lambda$$

$$+\frac{\kappa a_1(1-a_3)+a_2(1-a_3)-\sigma a_3}{\sigma\beta}.$$

The following proposition is proved in Barnett and Duzhak (2010) as Proposition 3.9.

Proposition 1.4.9. The New Keynesian model consisting of ((1.4.1), (1.4.2), (1.4.12)) produces a Hopf bifurcation at equilibrium points, if the transversality condition $\frac{\partial|\lambda_j(\mathbf{x}^*,\boldsymbol{\varphi}^*)|}{\partial\varphi_i^*}\Big|_{\varphi_i^*=\varphi_i^c} \neq 0$ holds, and if the parameters, $\boldsymbol{\varphi}^*$, satisfy the following three conditions at the equilibrium:

$$(a) \quad \left|\frac{\kappa a_1(1-a_3)+a_2(1-a_3)-\sigma a_3}{\sigma\beta}\right| < 1,$$

$$(b) \quad \left|\frac{\kappa a_1(1-a_3)+a_2(1-a_3)-\sigma a_3-\kappa-\sigma}{\sigma\beta}-1-a_3\right| < \frac{a_2\beta(a_3-1)+\kappa a_3+\sigma(1+a_3)}{\sigma\beta}+a_3,$$

$$(c) \quad a_3+\frac{a_2\beta(a_3-1)+\kappa a_3+\sigma(1+a_3)}{\sigma\beta}+\frac{((a_2+\kappa a_1)(1-a_3)-\sigma a_3)(\sigma\beta(1+a_3)+\kappa+\sigma)}{(\sigma\beta)^2}$$

$$= 1 - \left(\frac{(a_2 + \kappa a_1)(1 - a_3) - \sigma a_3}{\sigma\beta} \right)^2$$

Through numerical procedures, Barnett and Duzhak (2010) also find the existence of period-doubling bifurcation by varying a_2, while holding other parameters fixed in accordance with the appendix in Barnett and Duzhak (2010). The first period doubling bifurcation point is found at $a_2 = 5.7$. Starting from this point, Barnett and Duzhak (2010) then vary a_2 and a_3 simultaneously. They discover that period doubling bifurcation will occur for large values of the parameter a_2. As a result, aggressive reaction of the central bank to past values of the output gap can lead to a period doubling bifurcation within this model.

Duzhak (2010) started from point $a_2 = 5.7$ and varied parameters a_2 and a_1 simultaneously, while holding the other parameters constant in accordance with their paper's appendix. They numerically found a period doubling bifurcation boundary with values of the parameter a_2 within a very narrow range from 5.98 to 6.02. Barnett and Duzhak (2010) also found that a change in the interest rate smoothing parameter a_3 leads to a different critical period-doubling bifurcation value for the parameter a_2. Although previously thought to be the least prone to any kind of bifurcations, backward-looking interest rate rules show evidence of both Hopf bifurcation and period-doubling bifurcation.

x. Hybrid Rule With Interest Rate Smoothing Term

The hybrid rule with interest rate smoothing, proposed in Clarida, Gali and Gertler (1998), is often believed to match the empirics of Japan, the United States, and the European Union. That rule allows the central banker to set a short-term interest rate based on forecasted inflation, the current value of the output gap, and a past value of the interest rate, as follows:

$$i_t = (1 - a_3)(a_1 \pi_{t+1} + a_2 x_t) + a_3 i_{t-1}. \tag{1.4.13}$$

The model consisting of equation ((1.4.1), (1.4.2), (1.4.13)) can be written as

$$\mathbf{A} E_t \mathbf{x}_{t+1} = \mathbf{B} \mathbf{x}_t,$$

with

$$\mathbf{A} = \begin{bmatrix} 1 & \frac{1}{\sigma} & 0 \\ 0 & \beta & 0 \\ 0 & -a_1(1-a_3) & 1 \end{bmatrix}, \ \mathbf{B} = \begin{bmatrix} 1 & 0 & \frac{1}{\sigma} \\ -\kappa & 1 & 0 \\ a_2(1-a_3) & 0 & a_3 \end{bmatrix}, \mathbf{x}_t = \begin{bmatrix} x_t \\ \pi_t \\ i_{t-1} \end{bmatrix}.$$

This model has the following Jacobian:

$$\mathbf{J} = \begin{bmatrix} 1 + \frac{\kappa}{\sigma\beta} & -\frac{1}{\sigma\beta} & \frac{1}{\sigma} \\ -\frac{\kappa}{\beta} & \frac{1}{\beta} & 0 \\ \frac{a_1(1-a_3)\kappa}{\beta} + a_2(1-a_3) & \frac{a_1(1-a_3)}{\beta} & a_3 \end{bmatrix}$$

with characteristic polynomial

$$p(\lambda) = \lambda^3 - \left(1 + a_3 + \frac{\kappa}{\sigma\beta} + \frac{1}{\beta}\right)\lambda^2$$

$$+ \left(a_3 + \frac{1 + a_3}{\beta} - \frac{a_2(1-a_3)}{\sigma} + \frac{a_3\kappa + a_1\kappa(1-a_3)}{\sigma\beta}\right)\lambda$$

$$-\frac{a_3}{\beta} - \frac{a_3 a_2}{\sigma\beta} + \frac{a_2}{\sigma\beta}.$$

The following proposition is proved in Barnett and Duzhak (2010), Proposition 3.10.

Proposition 1.4.10. The New Keynesian model consisting of ((1.4.1), (1.4.2), (1.4.13)) produces a Hopf bifurcation at equilibrium points, if the transversality condition $\frac{\partial|\lambda_j(\mathbf{x}^*, \boldsymbol{\varphi}^*)|}{\partial \varphi_i^*}\Big|_{\varphi_i^*=\varphi_i^c} \neq 0$ holds, and if the parameters, $\boldsymbol{\varphi}^*$, satisfy the following three conditions at the equilibrium:

$$(a) \quad \left|-\frac{a_3}{\beta}-\frac{a_3 a_2}{\sigma\beta}+\frac{a_2}{\sigma\beta}\right|<1,$$

$$(b) \quad \left|\frac{a_2(1-a_3)-\kappa}{\sigma\beta}-1-a_3-\frac{1}{\beta}-\frac{a_3}{\beta}\right|<1+a_3+\frac{1+a_3}{\beta}-\frac{a_2(1-a_3)}{\sigma}+\frac{a_3\kappa+a_1\kappa(1-a_3)}{\sigma\beta},$$

$$(c) \quad a_3+\frac{1+a_3}{\beta}-\frac{a_2(1-a_3)}{\sigma}+\frac{a_3\kappa+a_1\kappa(1-a_3)}{\sigma\beta}$$

$$+\left(-\frac{a_3}{\beta}+\frac{a_2(1-a_3)}{\sigma\beta}\right)\left(1+a_3+\frac{1}{\beta}+\frac{\kappa}{\sigma\beta}\right)=1-\left(-\frac{a_3}{\beta}+\frac{a_2(1-a_3)}{\sigma\beta}\right)^2.$$

Through numerical procedures, Barnett and Duzhak (2010) find the existence of period-doubling bifurcation by varying a_2 while holding other parameters fixed in accordance with their appendix. The critical value of parameter a_2 is found at $a_2 = 3.03$. Starting with this point, Barnett and Duzhak (2010) first vary parameters a_2 and a_3 and then vary parameters a_2 and a_1 with the other parameters held constant.

In the first case, they find a fold flip bifurcation point at $a_2 = 3.03$ and $a_2 = 0.46$. In the second case, they find parameter a_2 is located mostly between 3 and 3.15 within the period-doubling bifurcation boundary, regardless of the values of parameter a_1. They conclude that a period doubling bifurcation will occur, if the central bank actively reacts to the output gap. Therefore, two types of bifurcations are revealed for the hybrid interest rate rule.

1.5 New Keynesian Model With Regime Switching[12]

1.5.1. Introduction

Monetary policy has seen major changes over the past decades. In the 1970s, the central bank stayed relatively passive in its actions in the presence of high inflation along with slow economic growth. Afterwards to help to combat high inflation present at the start of the 1980s, the Federal Reserve shifted to a more active regime. The phenomenon "great moderation" arose from the following period of moderate inflation along with stable economic growth in the mid-1980s. In the 21st century, following the financial crises starting in 2007, the Fed had to move aggressively.

Section 1.5, based on Barnett and Duzhak (2014), investigates whether bifurcations can result from monetary policy regime switching over time. Barnett and Duzhak (2014) focus on New Keynesian models. Previous literature like Gali and Gertler (1999), Bernanke, Laubach, Mishkin, and Posen (1999), and Leeper and Sims (1994) has shown that the original New Keynesian model has been developed into an important tool for monetary policy. In Barnett and Duzhak (2008) and Barnett and Duzhak (2010), the parameter space of the standard New Keynesian model has been shown to be stratified into bifurcation subsets. Relevant previous work includes, but is not limited to the following. Andrews (1993) and

[12] This section is summaried from Barnett and Duzhak (2014).

Evans (1985) study monetary policy with parameter instability. Davig and Leeper (2006) and Farmer, Waggoner, and Zha (2007) study determinacy when the Taylor rule is generalized to allow for regime switching. There is a literature on methods to determine parameter instability in time series (see Hansen (1992) and Nyblom (1989)). Economic models of regime switching had been investigated previously in different contexts, such as Hamilton (1989) and Warne (2000). Clarida, Gali, and Gertler (1999), Sims and Zha (2006), and Groen and Mumtaz (2008) find empirical support for regime switching in monetary policy.[13]

In Barnett and Duzhak (2014), the policy regime is assumed to follow a Markov chain with a fixed transition matrix. As a result, the solution to the model evolved differently depending on the state of the regime. Barnett and Duzhak (2014) investigate three models—a basic setup with a simple monetary policy rule, a New Keynesian model with regime switching, and a New Keynesian model with a hybrid monetary policy rule. They show through bifurcation analysis that regime switching can bring changes in the qualitative properties of the solution.

In the first model, the nominal interest rate is set as a function of current inflation with the response coefficient depending on the policy regime present at the time. Combining both the Fisher equation that links the nominal interest rate to future inflation, and the equation of real interest rate, Barnett and Duzhak (2014) get an equation that relates future inflation to current inflation and the real interest rate. A system of two linear difference equations is acquired for inflation in the two regimes. Barnett and Duzhak (2014) further use the eigenvalues of the system's matrix to perform the bifurcation analysis. Two main findings

[13] This model description is modified from Barnett and Duzhak (2014)

with respect to bifurcations are: first, for the basic setup, Barnett and Duzhak (2014) find no possibility of a Hopf bifurcation; second, they find the existence of a period doubling bifurcation. In this case, the solution can move from a stable to a periodic solution, where periodicity doubles in successive bifurcations.

In the second model, Barnett and Duzhak (2014) explore whether their analysis of this simple setup carries over to the standard New Keynesian model with regime switching and a standard Taylor rule. The Taylor (1999) rule makes the nominal interest rate a function of both inflation and the output gap. Barnett and Duzhak (2014) use numerical methods and find that this model does not exhibit any bifurcations for the range of feasible parameter combinations.

In the third model, Barnett and Duzhak (2014) investigate whether a state-of-the-art hybrid Taylor rule exhibits bifurcations. In this model, the Taylor rule allows for forward looking response to inflation. Using the same technique, they find that this model might exhibit a period-doubling bifurcation. The ideas from the basic setup thus carry over to the more prominent model of monetary policy. The analysis reveals that period doubling bifurcations and the resulting changes in the dynamics in inflation and output have more tendencies to arise in models with the forward-looking Taylor rule than in the model with the current-looking counterpart.

1.5.2. Dynamics with a Simple Monetary Policy Rule

The basic setup with simple monetary policy rule consists of the following two equations:

$$i_t = \alpha(s_t)\pi_t, \tag{1.5.1}$$

$$i_t = E_t\pi_{t+1} + r_t. \tag{1.5.2}$$

A policy reacts to inflation by changing an interest rate according to (1.5.1), where i_t is the nominal interest rate, $\alpha(s_t)$ a state-dependent coefficient which changes with the policy regime s_t, and π_t denotes the rate of inflation.

Under the assumption that there are two possible realizations for the policy regime, s_t, the linear reaction function to inflation evolves stochastically between two states, $s_t = 1$ and $s_t = 2$, so that

$$\alpha(s_t) = \begin{cases} \alpha_1 & \text{for } s_t = 1 \\ \alpha_2 & \text{for } s_t = 2, \end{cases}$$

where α_i denotes different parameters that govern the aggressiveness of policy to combat inflation. An active policy regime is the one with policy parameter $\alpha_i > 1$. In Barnett and Duzhak (2014), the active regime is regime 1. The policy regime evolves according to a Markov chain, where the transitional probabilities are given by the transition matrix with entries $p_{ij} = P[s_t = j|s_{t-1} = i]$ where $i, j = 1,2$.

Following Davig and Leeper (2006), Barnett and Duzhak (2014) use the Fisher equation (1.5.2) as the second equation in the model, where r_t is the real interest rate. The Fisher equation links the nominal interest rate to expected inflation and the real interest rate. Barnett and Duzhak (2014) use this relationship to solve for expected inflation, which evolves as a function of the nominal and real interest rates.

Combining (1.5.1) and (1.5.2), Barnett and Duzhak (2014) acquire the following dynamic system:

$$\begin{bmatrix} E_t[\pi_{1\,t+1}] \\ E_t[\pi_{2\,t+1}] \end{bmatrix} = \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix}^{-1} \begin{bmatrix} \alpha_1 & 0 \\ 0 & \alpha_2 \end{bmatrix} \begin{bmatrix} \pi_{1t} \\ \pi_{2t} \end{bmatrix} - \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix}^{-1} \begin{bmatrix} r_t \\ r_t \end{bmatrix}.$$

In this model, the real interest r_t is exogenously given. A fully specified macroeconomic model endogenizes this rate.

As is standard in the (bifurcation) analysis of difference equations, Barnett and Duzhak study the economy with parameter certainty. Parameter certainty in that model means that agents have no uncertainty about the level of inflation, if a certain state occurs. This does not mean agents know the level of inflation in the following period: the state of the policy regime determines inflation, and the state of the policy regime itself switches with given probabilities. Using parameter certainty, Barnett and Duzhak (2014) restate the system of linear difference equations as

$$\begin{bmatrix} \pi_{1\,t+1} \\ \pi_{2\,t+1} \end{bmatrix} = \begin{bmatrix} \dfrac{p_{22}\alpha_1}{p_{11}p_{22} - p_{12}p_{21}} & \dfrac{-p_{12}\alpha_2}{p_{11}p_{22} - p_{12}p_{21}} \\ \dfrac{-p_{21}\alpha_1}{p_{11}p_{22} - p_{12}p_{21}} & \dfrac{p_{11}\alpha_2}{p_{11}p_{22} - p_{12}p_{21}} \end{bmatrix} \begin{bmatrix} \pi_{1\,t} \\ \pi_{2\,t} \end{bmatrix}$$
$$- \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix}^{-1} \begin{bmatrix} r_t \\ r_t \end{bmatrix}.$$

Since the entries in the transition matrix are probabilities, it follows that $p_{11} + p_{21} = 1$ and $p_{22} + p_{12} = 1$. Hence, $\Delta = p_{11}p_{22} - p_{12}p_{21}$ as $= p_{11} + p_{22} - 1$.

To analyze the stability of the evolution of inflation and its dynamic properties, as shown by the linear system above, Barnett and Duzhak (2014) first consider the Jacobian matrix and corresponding characteristic polynomial of the above linear system:

$$\mathbf{J} = \begin{bmatrix} \dfrac{p_{22}\alpha_1}{p_{11}+p_{22}-1} & \dfrac{-p_{12}\alpha_2}{p_{11}+p_{22}-1} \\ \dfrac{-p_{21}\alpha_1}{p_{11}+p_{22}-1} & \dfrac{p_{11}\alpha_2}{p_{11}+p_{22}-1} \end{bmatrix}$$

$$P(\lambda) = \lambda^2 - b\lambda + c \text{ with } b = \frac{p_{22}\alpha_1 + p_{11}\alpha_2}{p_{11}+p_{22}-1} \text{ and } c = \frac{\alpha_1\alpha_2}{p_{11}+p_{22}-1}.$$

The determinant D of the Jacobian matrix is given by

$$D = \left[\frac{p_{22}\alpha_1 + p_{11}\alpha_2}{p_{11}+p_{22}-1}\right]^2 - \frac{4\alpha_1\alpha_2}{p_{11}+p_{22}-1}.$$

For a Hopf bifurcation to exist, the discriminant D must be negative, giving a rise to complex roots of $P(\lambda)$. Given that $(p_{11}+p_{22}-1)^2$ is always nonnegative, it follows that $D < 0$, which is equivalent to $(p_{22}\alpha_1 + p_{11}\alpha_2)^2 - (p_{11}+p_{22}-1)4\alpha_1\alpha_2 < 0$. The term on the left-hand side stays positive within the feasible set of parameters. Therefore, a Hopf bifurcation which arises only when the roots are complex, is not possible for this economy.

Barnett and Duzhak (2014) further examine the possibility of a period doubling bifurcation. Lemma 1 in Barnett and Duzhak (2014, page 10) provide conditions for the existence of the period doubling bifurcation (see Kuznetsov (1998), p.415). Both conditions for the period doubling bifurcation hold in this model. According to Barnett and Duzhak (2014), if one of the roots of the characteristic polynomial is in the negative part of the unit circle, there is a possibility of a period doubling bifurcation. They then analyze the eigenvalues of the characteristic polynomial. The characteristic polynomial $P(\lambda)$ has the following roots:

$$\lambda_{1,2} = \frac{1}{2}\left[\frac{\alpha_1 p_{22} + \alpha_2 p_{11}}{p_{11} + p_{22} - 1} \pm \sqrt{D}\right]$$

where D is the discriminant defined above.

According to Lemma 1 in Barnett and Duzhak (2014), they need one of the roots to be equal to -1. Setting $\lambda_{1,2} = -1$, the condition becomes

$$\sqrt{(p_{22}\alpha_1 + p_{11}\alpha_2)^2 - (p_{11} + p_{22} - 1)4\alpha_1\alpha_2} = 2(p_{11} + p_{22} - 1) + (p_{22}\alpha_1 + p_{11}\alpha_2),$$

which needs to hold for a period doubling bifurcation to occur. The above expression is simplified as

$$p_{11}(1 + \alpha_2) + p_{22}(1 + \alpha_1) + \alpha_1\alpha_2 = 1. \tag{1.5.3}$$

Equation (1.5.3) is a bifurcation boundary, in the form of a function of the parameters of the dynamical model.

To calibrate the economy, Barnett and Duzhak (2014) use the values in Table 1.5.1[14]. One of the policy regimes, regime 1, is active with a coefficient greater than 1, whereas regime 2 is a passive regime. They further assume that $p_{11} = 0$ is zero, which is the probability of remaining in the active regime, conditional on being in the active regime. Whenever regime 1 occurs, the economy will be sent to a passive regime with certainty.

[14] Table 1.5.1 is a replicate of Barnett and Duzhak's (2014) Table 1.

Table 1.5.1 Standard Parameter Combinations

Parameter	Value
α_1	1.5
α_2	0.5
γ_1	0.3
γ_2	0.15
p_{11}	0.85
p_{22}	0.9
β	0.98
κ	0.024
σ	0.3

Using these assumptions, Barnett and Duzhak (2014) find the critical value for the transitional probability p_{22} to be $p_{22}^c = 0.1$. They use this point as a benchmark to trace out the bifurcation boundary. Varying the other parameters, i.e. policy parameters α_1 and α_2, along with the probability of staying in the passive regime p_{22}, Barnett and Duzhak (2014) demonstrate a period doubling bifurcation boundary as a function of the three control parameters p_{22}, α_1, and α_2. If $p_{22} = 1$, then the policy regime would be passive and stay passive indefinitely. In this case, $1 + \alpha_1 + \alpha_1\alpha_2 = 1$, so no bifurcation can arise. If $p_{22} = 0$, then $\alpha_1\alpha_2 = 1$. The bifurcation boundary is symmetric with respect to the policy parameters α_1 and α_2. If the policy reaction coefficient α_2 of the passive regime is small, the policy response coefficient α_1 needs to be very large for a bifurcation to arise.

1.5.3 New Keynesian Model with Regime Switching

The standard New Keynesian model, as laid out in, e.g., Woodford (2003) or Walsh (2003), traditionally consists of the following equations:

$$x_t = E_t x_{t+1} - \frac{1}{\sigma}(i_t - E_t \pi_{t+1}) + u_t^D \qquad (1.5.4)$$

$$\pi_t = \beta E_t \pi_{t+1} + \kappa x_t + u_t^S. \qquad (1.5.5)$$

$$i_t = \alpha(s_t)\pi_t + \gamma(s_t)x_t \qquad (1.5.6)$$

Equation (1.5.4) is the forward-looking IS equation describing the demand side of the economy ,and equation (1.5.5) is the Phillips curve representing the supply side. The IS curve (1.5.4) relates the output gap, x_t, to the nominal interest rate, i_t, and expectations about the future output gap as well as inflation. The coefficient $\frac{1}{\sigma}$ is the inverse of relative risk aversion, which equals the elasticity of intertemporal substitution, since preferences with constant relative risk aversion are assumed in deriving the equations. The New Keynesian Phillips curve, (1.5.5), describes how inflation is driven by the output gap and expected inflation. Both equations for demand and supply side allow for a shock, u_t. A rule for monetary policy is (1.5.6), which takes the form described in Taylor (1999). According to that Taylor rule, the monetary authority sets the nominal interest rate by targeting both inflation and the output gap, where α_i governs the Central bank's reaction to inflation and γ_i the reaction to the output gap.

The model can be written in matrix notation

$$\mathbf{AY_{t+1}} = \mathbf{BY_t} + \mathbf{u_t}, \qquad (1.5.7)$$

where **Y** denotes the vector of variables $\mathbf{Y} = [\pi_{1t}\ \pi_{2t}\ x_{1t}\ \ x_{2t}]^T$ and $\mathbf{u_t}$ the vector of aggregate demand and supply shocks, while **A** and **B** are given by

$$\mathbf{A} = \begin{bmatrix} \beta p_{11} & \beta(1-p_{22}) & 0 & 0 \\ \beta(1-p_{11}) & \beta p_{22} & 0 & 0 \\ \frac{p_{11}}{\sigma} & \frac{1-p_{22}}{\sigma} & p_{11} & 1-p_{22} \\ \frac{1-p_{11}}{\sigma} & \frac{p_{22}}{\sigma} & 1-p_{11} & p_{22} \end{bmatrix},$$

and

$$\mathbf{B} = \begin{bmatrix} 1 & 0 & -\kappa & 0 \\ 0 & 1 & 0 & -\kappa \\ \frac{\alpha_1}{\sigma} & 0 & 1+\frac{\gamma_1}{\sigma} & 0 \\ 0 & \frac{\alpha_2}{\sigma} & 0 & 1+\frac{\gamma_2}{\sigma} \end{bmatrix}.$$

Rearranging (1.5.7), Barnett and Duzhak (2014) obtain the normal form

$$\mathbf{Y_{t+1}} = \mathbf{CY_t} + \mathbf{A^{-1}u_t}, \qquad (1.5.8)$$

where $\mathbf{C} = \mathbf{A^{-1}B}$.

Now the system is 4-dimensional instead of having a two-by-two Jacobian matrix in the basic form. Since the 4-dimensional model is more difficult to analyze, Barnett and Duzhak (2014) employ the software continuation package CONTENT developed by Yuri Kuznetsov and V.V.Levitin to trace out bifurcation boundaries. Barnett and Duzhak (2014) hold constant the parameters that describe the probabilities of regime, while varying structural and policy parameters. They find that neither a Hopf nor a periodic doubling bifurcation can occur for any feasible set of parameters. They do find a bifurcation for parameter values $\gamma_2 = 0.179$ and $\kappa = -0.46$. However, negative values for κ are economically nonfeasible. In this case, the bifurcation boundary never crosses into the subspace of feasible parameter combinations.

1.5.4 New Keynesian Model with a Hybrid Monetary Policy Rule

Barnett and Duzhak (2014) further provide an analysis of a state-of-the-art model of a monetary policy. Proposed by Clarida, Gali and Gertler (1999), the model consists of a hybrid rule, which includes both a current-looking and a forward-looking component:

$$i_t = \alpha(s_t)\pi_{t+1} + \gamma(s_t)x_t. \tag{1.5.9}$$

According to the rule, a policy maker is forward-looking with respect to inflation and current looking with respect to the output gap. The corresponding linear system is:

$$\mathbf{Y_{t+1}} = \mathbf{DY_t},$$

where matrix **D** is given by

$$\mathbf{D} =$$

$$\begin{bmatrix} \frac{p_{22}}{\beta(-1+p_{22}+p_{11})} & \frac{-1+p_{22}}{\beta(-1+p_{22}+p_{11})} & \frac{-p_{22}k}{\beta(-1+p_{22}+p_{11})} & -\frac{(-1+p_{22})k}{\beta(-1+p_{22}+p_{11})} \\ \frac{-1+p_{11}}{\beta(-1+p_{22}+p_{11})} & \frac{p_{11}}{\beta(-1+p_{22}+p_{11})} & -\frac{(-1+p_{11})k}{\beta(-1+p_{22}+p_{11})} & -\frac{p_{11}k}{\beta(-1+p_{22}+p_{11})} \\ \frac{p_{22}(-1+\alpha_1)}{\sigma\beta(-1+p_{22}+p_{11})} & \frac{(-1+p_{22})(-1+\alpha_2)}{\sigma\beta(-1+p_{22}+p_{11})} & \frac{p_{22}(k-k\alpha_1+\sigma\beta+\beta\gamma_1)}{\sigma\beta(-1+p_{22}+p_{11})} & \frac{(-1+p_{22})(k-k\alpha_2+\sigma\beta+\beta\gamma_2)}{\sigma\beta(-1+p_{22}+p_{11})} \\ \frac{(-1+p_{11})(-1+\alpha_1)}{\sigma\beta(-1+p_{22}+p_{11})} & \frac{(-1+\alpha_2)p_{11}}{\sigma\beta(-1+p_{22}+p_{11})} & \frac{(-1+p_{11})(k-k\alpha_1+\sigma\beta+\beta\gamma_1)}{\sigma\beta(-1+p_{22}+p_{11})} & \frac{p_{11}(k-k\alpha_1+\sigma\beta+\beta\gamma_2)}{\sigma\beta(-1+p_{22}+p_{11})} \end{bmatrix}$$

Numerical analysis of this dynamic system to find Hopf and period doubling bifurcations leads to two findings, which are the same as for the simple economy. First, there is no possibility of a Hopf bifurcation. Second, a period doubling bifurcation emerges.

To find a bifurcation boundary, Barnett and Duzhak (2014) first vary parameter α_2, while holding all other parameters constant. They use the critical point of α_2 at 0.00125 to trace out the bifurcation boundary. After tracking the first period doubling bifurcation point, Barnett and Duzhak (2014) choose the second parameter, the risk aversion parameter, σ, to vary simultaneously with parameter α_2. They find a period doubling bifurcation will occur for a very narrow set of parameters α_2 corresponding to a passive reaction to future inflation, in the close proximity of zero. Similarly, they find a period doubling point for parameter $\kappa = 3.725$. After choosing a second parameter, σ, to be varied, Barnett and Duzhak (2014) compute the period doubling bifurcation boundary. Parameter κ is a nonlinear function of the discount factor and the parameter responsible for the degree of price rigidity. It shows that the period doubling bifurcation will occur, when the economy is characterized by a high level of price stickiness. After analyzing further parameter combinations, Barnett and Duzhak (2014) find that a period doubling bifurcation is also possible for lower values of κ accompanied by very high values of the policy parameter, α_1, which shows that an aggressive reaction of the central bank to future inflation will lead to a period doubling bifurcation.

1.6 Zellner's Marshallian Macroeconomic Model[15]

1.6.1 Introduction

This section describes Banerjee, Barnett, Duzhak, and Gopalan's (2011) bifurcation analysis of the Marshallian Macroeconomic Model. The Marshallian Macroeconomic Model (MMM) in Zellner and Israilevich (2005) is described by sectoral demand, supply, and entry/exit equations, as well as factor markets, the government, and a monetary sector added to complete the model. The explicitly formulated entry/exit behavior model in the MMM can

[15] This section is summarized from Banerjee, Barnett, Duzhak and Gopalan (2011).

be described by the equation $\frac{\dot{N}}{N} = \gamma'(\Pi - F^e)$; i.e. the growth rate of firms in the industry is proportional to the difference in current industry profitability, Π, and the long-run future profitability in the industry, F^e. The speed of adjustment is determined by the parameter γ'. With an entry/exit equation for each industry introduced in the model, Zellner and Israilevich (2005) describe the dynamics of the model in key variables, such as price and output at the sectoral as well as at the aggregate level. Varying some parameters would change the equilibria and could possibly cause changes in the nature of the equilibria, such as the number of solutions and the stability properties of the equilibria. Banerjee, Barnett, Duzhak, and Gopalan (2011) examine the model's characteristics, as well as the possibility of cyclical behavior through bifurcation analysis with respect to the entry/exit parameter F^e.

Banerjee, Barnett, Duzhak, and Gopalan (2011) show that a Hopf bifurcation exists within the theoretically feasible parameter space, giving rise to stable cycles, when taking F_1 from the entry-exit equation as the candidate for bifurcation parameter. Future work with that model could take several directions. One would be to introduce expectations into firms' future profitability. Another could be to introduce the money market and examine the possibility of other kinds of bifurcations with respect to government and monetary policy parameters.

1.6.2 The Model[16]

Banerjee, Barnett, Duzhak, and Gopalan (2011) consider a two sector, continuous time version of the Marshallian Macroeconomic Model (MMM) as outlined in Zellner and Israilevich (2005). Each sector is characterized by an aggregate output demand function, an

[16] The model description is modified from Banerjee, Barnett, Duzhak, and Gopalan (2011).

aggregate supply function, and entry-exit modeling. Banerjee, Barnett, Duzhak, and Gopalan (2011) also include the government that collects taxes on output, purchases output from the two sectors and inputs from the factor markets. They exclude the presence of money markets from the model at this stage.

i. Ouput Demand

As noted in Banerjee, Barnett, Duzhak, and Gopalan (2011), the total demand for goods in the ith sector, i =1,2, is the sum of the demands from the government and the aggregate demand from households. Aggregate demand is thus given by

$$S_i = G_i + P_i^{1-\eta_{ii}} P_j^{\eta_{ij}} (S(1 - T^s))^{\eta_{is}}, \tag{1.6.1}$$

where G_i is the nominal government expenditure in sector i, $S = S_1 + S_2$ is the total income (nominal output), T^s is the tax rate, η_{ii} is the own price elasticity, η_{ij} is the cross price elasticity, and η_{is} is the income elasticity.

To express (1.6.1) in terms of growth rates, the aggregate demand for goods in each sector is the weighted sum of growth rates of demand from the government and households,

$$\hat{S}_i = g_i \hat{G}_i + (1 - g_i)\left[(1 - \eta_{ii})\hat{P}_i + \eta_{ij}\hat{P}_j + \eta_{is}\left(\hat{S} + \hat{T}^{s\prime}\right)\right], \tag{1.6.2}$$

where g_i is the ratio of government spending in sector i to total sales in sector i and $T^{s\prime} = 1 - T^s$. We use the hat over symbols to designate growth rate.

ii. Output Supply

There are N_i identical firms in the ith sector, each using a Cobb-Douglas type production function, $q_i = A_i^* L_i^{\alpha} K_i^{\beta}$, with $0 < \alpha_i, \beta_i < 1$, and $0 < \theta_i = 1 - \alpha_i - \beta_i < 1$, where q_i is the product of a neutral technological change, labor, and capital augmentation factors. The aggregate nominal profit-maximizing output supply of each sector i is given by $S_i = N_i P_i^{\frac{1}{\theta_i}} w^{-\frac{\alpha_i}{\theta_i}} r^{-\frac{\beta_i}{\theta_i}}$, where P_i, ω, and r are the price, wage rate, and rental rate respectively. Converting to growth rates, output supply becomes

$$\hat{S}_i = \hat{N}_i + \frac{1}{\theta_i}\hat{P}_i - \frac{\alpha_i}{\theta_i}\hat{\omega} - \frac{\beta_i}{\theta_i}\hat{r}. \tag{1.6.3}$$

iii. Entry/Exit

Banerjee, Barnett, Duzhak, and Gopalan (2011) consider the simplest form of the entry/exit equation proposed by Zellner and Israilevich (2005),

$$\hat{N}_i = \gamma_i[\Pi_i - F_i], \tag{1.6.4}$$

where $\Pi_i = \theta_i S_i$ is the current nominal aggregate industry profit for sector i, while $F_i > 0$ represents the aggregate long-run equilibrium profits in sector i, taking account of discounted entry costs. These parameters are considered by Banerjee, Barnett, Duzhak, and Gopalan (2011) to be time invariant. The coefficient, $\gamma_i > 0$, is the speed of adjustment for sector i. The larger the value of γ_i, the faster the adjustment is.

The interpretation of the entry/exit equation in Banerjee, Barnett, Duzhak, and Gopalan (2011) is that a positive departure from equilibrium profits F_i^e will attract new firms into the industry, while a negative departure will induce firms to leave the industry, given $\gamma_i > 0$.

iv. Government

According to Banerjee, Barnett, Duzhak, and Gopalan (2011), total nominal government expenditure, G, is the sum of expenditures in each of the two sectors, G_i, and its expenditure on labor, G_L, and capital, G_K. Zellner and Israilevich (2005) assume that G_i, for all $i = 1,2, L, K$, grows at the same rate as G. Under this assumption, Banerjee, Barnett, Duzhak, and Gopalan (2011) propose that $G_i = \zeta_i G$, where ζ_i is the fraction of total government expenditure in the ith market. Thus in terms of growth rates, we have $\hat{G}_i = \hat{G}$.

The government collects a single uniform tax at the rate T^s on output. The tax revenue R is given by $R = T^s S$, which is expressed as $\hat{R} = \hat{T}^s + \hat{S}$ in terms of growth rate. The exogenously determined deficit/surplus, D, is defined as the government expenditures as a percentage of revenues, i.e. $D = \frac{G}{R}$. In terms of growth rate, we have

$$\hat{G} = \hat{D} + \hat{R} = \hat{D} + \hat{T}^s + \hat{S}. \tag{1.6.5}$$

v. Factor Markets

According to Banerjee, Barnett, Duzhak, and Gopalan (2011), the aggregate profit-maximizing factor demands from sector i are $L_i = \frac{\alpha_i s_i}{\omega}$ and $K_i = \frac{\beta_i s_i}{r}$. The government demand for labor and capital are $L_g = \frac{G_L}{\omega}$ and $K_g = \frac{G_K}{r}$ respectively. In terms of growth rates, the total demand for each factor is the weighted sum of growth rates of sectoral demands and the government demand for that factor, shown as below:

$$\frac{L_1}{L}\hat{L}_1 + \frac{L_2}{L}\hat{L}_2 + \frac{L_g}{L}\hat{L}_g = l_1\hat{L}_1 + l_2\hat{L}_2 + l_g\hat{L}_g, \tag{1.6.6}$$

$$\frac{K_1}{K}\hat{K}_1 + \frac{K_2}{K}\hat{K}_2 + \frac{K_g}{K}\hat{K} = k_1\hat{K}_1 + k_2\hat{K}_2 + k_g\hat{K}_g. \tag{1.6.7}$$

The dependence of the weights is given in Appendix A in Banerjee, Barnett, Duzhak, and Gopalan (2011). According to Zellner and Israilevich (2005), $L = (\frac{\omega}{P})^{\delta}(\frac{S}{P})^{\delta_s}$ and $K = (\frac{r}{P})^{\phi}(\frac{S}{P})^{\phi_s}$, where δ (or ϕ) and δ_s (or ϕ_s) are price and income elasticities of labor (or capital). In terms of growth rates, the labor and capital supplies equal

$$\hat{L} = \delta(\hat{\omega} - \hat{P}) + \delta_s(\hat{S} - \hat{P}), \tag{1.6.8}$$

$$\hat{K} = \phi(\hat{r} - \hat{P}) + \phi_s(\hat{S} - \hat{P}). \tag{1.6.9}$$

vi. Quantity and Price Aggregates

The growth rates of aggregate nominal sales and the price aggregate are given by

$$\hat{S} = s_1\hat{S}_1 + s_2\hat{S}_2, \tag{1.6.10}$$

$$\hat{P} = s_1\hat{P}_1 + s_2\hat{P}_2, \tag{1.6.11}$$

where $s_i = \frac{S_i}{S}$.

vii. Solving the Model

The MMM model is solved using market clearing conditions in all markets and the government's flow budget identity. The complete solution procedure is outlined in Appendix A in Banerjee, Barnett, Duzhak, and Gopalan (2011). All the equations in the model are reduced to yield the following two dynamic equations that govern the behavior of S_1 and S_2:

$$\begin{bmatrix} \dot{S}_1 \\ \dot{S}_2 \end{bmatrix} = \begin{bmatrix} \mathcal{F}_1(S_1, S_2; \mathbf{\Omega}) \\ \mathcal{F}_2(S_1, S_2; \mathbf{\Omega}) \end{bmatrix} = \boldsymbol{\mathcal{F}}(S_1, S_2; \mathbf{\Omega}). \tag{1.6.12}$$

The explicit form of the non-linear functions, $\mathcal{F}_1$ and $\mathcal{F}_2$, can be found in Appendix A in Banerjee, Barnett, Duzhak, and Gopalan (2011). The vector $\mathbf{\Omega}$ consists of all structural parameters. The entry parameter for sector 1, F_1, is taken as the bifurcation parameter in the following section. According to Appendix A in Banerjee, Barnett, Duzhak, and Gopalan (2011),

$$\boldsymbol{\mathcal{F}}(S_1, S_2; \boldsymbol{\Omega}) = (\boldsymbol{\mathcal{H}}(S_1, S_2; \boldsymbol{\Omega}))^{-1}\boldsymbol{\mathcal{D}}(S_1, S_2; \boldsymbol{\Omega}), \tag{1.6.13}$$

where $\boldsymbol{\mathcal{H}}$ is a matrix of dimension 2×2 and $\boldsymbol{\mathcal{D}}$ is a vector of dimension 2×1. The elements of $\boldsymbol{\mathcal{H}}$ and $\boldsymbol{\mathcal{D}}$ produce a high degree of nonlinearity in $\boldsymbol{\mathcal{F}}$**.** **I**n determining the dynamics of the equilibrium, several equilibria can arise.

To solve for an equilibrium, (S_1, S_2), such that $\dot{S}_1 = 0$ and $\dot{S}_2 = 0$, it suffices to solve $\boldsymbol{\mathcal{F}}(S_1, S_2; \boldsymbol{\Omega}) = 0$ in the system (6.12). From equation (6.13), the solutions at which $\boldsymbol{\mathcal{D}} = 0$ will always be an equilibrium. Assuming there is no growth in government deficit, D, and taxes, T^s, the solution is based on (1.6.4), so that

$$S_1 = \frac{1}{\theta_1}F_1 \text{ and } S_2 = \frac{1}{\theta_2}F_2\,. \tag{1.6.14}$$

The positive solutions are economically relevant and produce long run equilibrium by ensuring that there is no further entry/exit in either sector. The next section surveys Banerjee, Barnett, Duzhak, and Gopalan's (2011) results on stability and their bifurcation analysis of this equilibrium.

1.6.3 Stability and Bifurcation Analysis of Equilibrium

By generalizing the analysis of Veloce and Zellner's (1985) one sector MMM model to two sectors, Banerjee, Barnett, Duzhak, and Gopalan (2011) analyze the dynamics in terms of convergence to the equilibrium given by (1.6.14). They consider the effects of cross price and income elasticities along with own price elasticities and emphasize two results that arise in the multisector model: (1) the solution may be stable, even when the two sectors have elastic demand; and (2) the path to the long run equilibrium may not be monotonic, so oscillatory damped convergence may arise.

Banerjee, Barnett, Duzhak, and Gopalan (2011) explain the occurrence of oscillatory convergence to equilibrium in terms of economic theory. They begin the analysis by assuming that the two sectors produce normal goods, which are substitutes and have elastic demand, and assuming Sector 1 is out of equilibrium, so that $S_1 > \frac{1}{\theta_1} F_1$, and $S_2 = \frac{1}{\theta_2} F_2$. Since $S_1 > \frac{1}{\theta_1} F_1$, current profitability is higher than equilibrium profitability, so entry takes place in Sector 1. The increase of supply in Sector 1 causes a drop in Sector 1's price, P_1, and consequently causes sales, S_1, having elastic demand, to increase. In addition, there is a decrease in Sector 2's demand, since the two goods are substitutes. There are two opposing effects on S_1. If Sector 2's demand decreases, both Sector 2's price, P_2, and quantity, Q_2, decline, leading to a decline in Sector 2's sales, S_2. If this decline in S_2 is greater in magnitude than the initial increase in S_1, then $S = S_1 + S_2$ will decline, resulting in a fall in S_1. Hence cross price and aggregate income effect may offset, having potentially destabilizing influence.

Banerjee, Barnett, Duzhak, and Gopalan (2011) further note that the decline in P_1 causes a decrease in Sector 2's demand and hence a decline in Sector 2's sales, which drop below the equilibrium, so that $S_2 < \frac{1}{\theta_2} F_2$. The result is an increase in S_2 and consequently an increase in S_1 through the income effect. Consequently the oscillatory convergence to equilibrium arises from interaction between the magnitudes of the shift and the elasticities. The mechanism depends largely on the own price, cross price, and income elasticities, and the magnitude of the shifts in demand and supply in each sector. Banerjee, Barnett, Duzhak, and Gopalan (2011) observe it is possible that the insufficiency of these shifts may result in the unstable solution, and they emphasize the importance of consistency between the elasticity parameters and the values of other parameters in production, input markets, entry/exit equations, and government policy. The possibility exists that the economy could change its convergence type, if some of these parameters were to change.

Banerjee, Barnett, Duzhak, and Gopalan (2011) find the existence of a Hopf bifurcation, occurring when the Jacobian of $\boldsymbol{\mathcal{F}}$ has a pair of purely imaginary eigenvalues at some critical value of a bifurcation parameter. In the following analysis, they vary only parameter F_1, while keeping all other parameters at values given in their paper's Appendix B. To analyze a codimension-1 Hopf bifurcation for the system (1.6.12), they first search for the value of (S_1, S_2) and the bifurcation parameter (F_1) satisfying the following conditions:

$$\mathcal{F}_1(S_1, S_2, F_1) = 0, \tag{1.6.15}$$

$$\mathcal{F}_2(S_1, S_2, F_1) = 0, \tag{1.6.16}$$

$$tr(\mathbf{J}_{\mathcal{F}}(S_1, S_2, F_1)) = 0, \tag{1.6.17}$$

$$det\big(\mathbf{J}_{\mathcal{F}}(S_1, S_2, F_1)\big) > 0, \tag{1.6.18}$$

where $\mathbf{J}_{\mathcal{F}}$ is the Jacobian of $\boldsymbol{\mathcal{F}}$.

Banerjee, Barnett, Duzhak, and Gopalan (2011) observe that equations (1.6.15) and (1.6.16) yield the equilibrium for the system of differential equations in (1.6.12). Conditions (1.6.17) and (1.6.18) ensure that the eigenvalues of $\mathbf{J}_{\mathcal{F}}$ are purely imaginary. They find the existence of a Hopf bifurcation at the computed critical value $F^H = 6.070386762$ by verifying that conditions (1.6.17) and (1.6.18) are satisfied and the slope of the trace is not zero. Thus, as the parameter F_1 crosses F^H from the right, the solution given in (1.6.14) goes from a stable equilibrium to an unstable one. Banerjee, Barnett, Duzhak, and Gopalan (2011) illustrate that the system is locally spiraling inward for $F_1 > F^H$, and the system exhibits stable cycles in the phase space for F_1 close enough to $F^H$and $F_1 < F^H$.

1.7 Open-Economy New Keynesian Models[17]

1.7.1 Introduction

The Barnett and Duzhak's (2008, 2010, 2013) results surveyed in sections 4 and 5 on bifurcation of New Keynesian models is based on closed economy models. Continuing to explore bifurcation in macroeconometric models, Barnett and Eryilmaz (2014) explore bifurcation of an open economy New Keynesian model proposed by Gali and Monacelli (2005). In addition, Barnett and Eryilmaz (2013) explore bifurcation of the open economy New Keyensian model proposed by Claridy, Gali, and Gertler (2002). In this section, we first survey the results of Barnett and Eryilmaz (2014) and then the results of Barnett and Eryilmaz (2013).

With those two models, Barnett and Eryilmaz (2013, 2014) find that the open economy framework has more complex dynamics than the closed economy models. As a result, stratification of the confidence regions remains an important research topic in the context of open-economy New Keynesian structures. In addition to damaging inference

[17] This section is summarized from Barnett and Eryilmaz (2013,2014).

robustness, bifurcation of those models can result from changes in monetary policy. Such phenomena are relevant to evaluating policy risk.

As surveyed in section 1.7.2 below, Barnett and Eryilmaz (2014) ran bifurcation analyses of the Gali and Monacelli (2005) model and found that the degree of openness has a significant role in equilibrium determinacy and emergence of bifurcations. The values of bifurcation parameters and location of bifurcation boundaries are affected by introducing parameters related to the open economy structure. Numerical analyses are performed to search for different types of bifurcation. Limit cycles and period doubling bifurcations are found, although in some cases only for nonfeasible parameter values. Stratification of the confidence regions remains problematic to open economy New Keynesian functional structures.

Comparing the results from Barnett and Duzhak's (2010) closed economy analysis, it is not clear whether openness makes the New Keynesian model more sensitive to bifurcations. Barnett and Eryilmaz (2014) do not find evidence that open economies are more vulnerable to the problem than closed economies. The evidence from the Gali and Monacelli model might be caused by the model's broad set of parameters, including deep parameters relevant to the open economy. The fact that the studies use different sets of benchmark values for the parameters makes direct comparison more difficult. Barnett and Eryilmaz (2014) also note that the analysis is restricted to special cases within the framework of open-economy New Keynesian structures. Generalizing the results to real economies would require more results with other open-economy New Keynesian models.

As surveyed in section 1.7.3 below, Barnett and Eryilmaz (2013) investigate bifurcations in the Clarida, Gali, and Gertler (2002) model. Barnett and Eryilmaz (2013) show that the model is vulnerable to Hopf bifurcation at a critical value of the parameter measuring the sensitivity of the nominal interest rate to changes in output gap. Their theoretical results need to be confirmed by subsequent numerical analysis to locate the Hopf bifurcation boundary and map its shape. The numerical analysis is beyond the scope of Barnett and Eryilmaz (2013), but they have provided the theory needed to implement the numerical research and locate the Hopf bifurcation boundary. A primary objective of the subsequent numerical analysis should be to determine whether the Hopf bifurcation boundary crosses relevant confidence regions of the model's parameters. If so, a serious robustness problem would exist in dynamical inferences. But even if the bifurcation boundary does not cross the confidence region, policy can move the location of the bifurcation boundary. Within this model, the central bank should react cautiously to changes in the rate of domestic inflation and the output gap to avoid inducing instability from a possible Hopf bifurcation.

1.7.2 Gali and Monacelli Model[18]

The Gali and Monacelli (2005) model is described by the following equations:

$$x_t = E_t x_{t+1} - \frac{1+\alpha(\omega-1)}{\sigma}(r_t - E_t \pi_{t+1} - \bar{r}_t), \tag{1.7.1}$$

$$\pi_t = \beta E_t \pi_{t+1} + \frac{(1-\beta\theta)(1-\theta)}{\theta}\left(\frac{\sigma}{1+\alpha(\omega-1)} + \varphi\right)x_t, \tag{1.7.2}$$

$$r_t = \overline{r_t} + \phi_\pi \pi_t + \phi_x x_t. \tag{1.7.3}$$

The Gali and Monacelli (2005) model is based on the following assumptions: the domestic policy does not affect the other countries or the world economy; each economy is

[18] The model description is modified from Barnett and Eryilmaz (2014).

assumed to have identical preferences, technology, and market structure; both consumers and firms are assumed to behave optimally. Consumers maximize expected present value of utility, while firms maximize profits.

The utility maximization problem yields the dynamical intertemporal IS curve (1.7.1), which is a log-linear approximation to the Euler equation. In equation (1.7.1), x_t is the gap between actual output and flexible-price equilibrium output, $\bar{r}_t$ is the small open economy's natural rate of interest, and $\sigma_\alpha = \sigma(1 - \alpha + \alpha\omega)^{-1}$ and $\omega = \sigma\gamma + (1 - \alpha)(\sigma\eta - 1)$ are composite parameters. The lowercase letters denote the logs of the respective variables, $\rho = \beta^{-1} - 1$ denote the time discount rate, and α is the log of labor's average product. The maximization problem of the representative firm yields the aggregate supply curve (1.7.2), also often called the New Keynesian Philips curve in log-linearized form.

The policy rule (1.7.3) is a version of the Taylor rule, providing a simple (non-optimized) monetary policy, where the coefficients $\phi_x > 0$ and $\phi_\pi > 0$ measure the sensitivity of the nominal interest rate to changes in output gap and inflation rate respectively. Various versions of the Taylor rule are often employed to design monetary policy in empirical DSGE models. Equations (1.7.1) and (1.7.2), in combination with a monetary policy rule such as equation (1.7.3), constitute a small open economy model in the New Keynesian tradition.

Gali and Monacelli (2005) observed that closed economy models and open economy models differ in two primary aspects: (1) some coefficients, such as the degree of openness, terms of trade, and substitutability among domestic and foreign goods, depend on the parameters that are exclusive to the open economy framework; and (2), the natural levels of output and interest rate depend upon both domestic and foreign disturbances, in addition to openness and terms of trade. Barnett and Eryilmaz (2014) use the same methodology as in section 1.4 to detect bifurcation phenomenon. For two-dimensional dynamical systems, they apply Theorem 1.1.1. For three-dimensional dynamical systems, they apply Theorem 1.4.1. They employed CL MatCont within MatLab for numerical analysis. Regarding different policy rules, Barnett and Eryilmaz (2014) consider contemporaneous, forward, and backward looking policy rules, as well as hybrid combinations. The calibrated values of the parameters are given in Gali and Monacelli (2005), which are $\beta = 0.99, \alpha = 0.4, \sigma = \omega = 1, \varphi = 3,$ and $\mu = 0.086$. For the $N = 3$ policy parameters, $\phi_x = 0.125, \phi_\pi = 1.5$, and $\phi_r = 0.5$.

i. Current-Looking Taylor Rule

The model consists of the following equations, in which the first two equations describe the economy, while the third equation is the monetary policy rule followed by the central bank with $N = 2$ policy parameters:

$$\pi_t = \beta E_t \pi_{t+1} + \mu \left(\frac{\sigma}{1+\alpha(\omega-1)} + \varphi \right) x_t, \tag{1.7.4}$$

$$x_t = E_t x_{t+1} - \frac{1+\alpha(\omega-1)}{\sigma} (r_t - E_t \pi_{t+1} - \bar{r}_t), \tag{1.7.5}$$

$$r_t = \bar{r}_t + \phi_\pi \pi_t + \phi_x x_t. \tag{1.7.6}$$

Rearranging the terms, the system can be written in the form $E_t \mathbf{y_{t+1}} = \mathbf{C} \mathbf{y_t}$,

$$\begin{bmatrix} E_t x_{t+1} \\ E_t \pi_{t+1} \end{bmatrix} = \begin{bmatrix} 1+\dfrac{\mu}{\beta}+(1+\alpha(\omega-1))\left(\dfrac{\beta\phi_x+\varphi\mu}{\beta\sigma}\right) & \dfrac{(\beta\phi_x-1)(1+\alpha(\omega-1))}{\beta\sigma} \\ -\dfrac{\mu}{\beta}\left(\varphi+\dfrac{\sigma}{1+\alpha(\omega-1)}\right) & \dfrac{1}{\beta} \end{bmatrix} \begin{bmatrix} x_t \\ \pi_t \end{bmatrix}.$$

(1.7.7)

Using Theorem 1.1.1, the conditions for the existence of Hopf bifurcation in the system (1.7.7) are presented in the following proposition.

Proposition 1.7.1 Let Δ be the discriminant of the characteristic equation. Then system (1.7.7) undergoes a Hopf bifurcation at equilibrium points, if and only if $\Delta < 0$ and

$$\phi_x^* = \frac{\sigma(\beta-1)}{1+\alpha(\omega-1)} - \mu(\varphi+\frac{\sigma}{1+\alpha(\omega-1)})\phi_\pi. \tag{1.7.8}$$

The corresponding value of the bifurcation parameter in the closed economy case is $\phi_x^* = \sigma(\beta-1) - \kappa\phi_\pi$, as given by Barnett and Duzhak (2008). For $\alpha = 0$, proposition 1.7.1 gives the same result as the closed economy counterpart.

Barnett and Eryilmaz (2014) numerically find a period doubling bifurcation at $\phi_x = -2.43$ and a Hopf bifurcation at $\phi_x = -0.52$. Numerical computations indicate that the monetary policy rule equation (1.7.6) should have $\phi_x^* < 0$ for a Hopf or period doubling bifurcation to occur. That negative coefficient for the output gap in equation (1.7.6) would indicate a procyclical monetary policy: rising interest rates, when the output gap is negative, or vice versa. Literature seeking to explain procyclicality in monetary policy includes Schettkat and Sun (2009), Demirel (2010), and Leith, and Moldovan, and Rossi (2009). A successful countercyclical monetary policy would be bifurcation-free and would yield more robust dynamical inferences with confidence regions not crossing a bifurcation boundary.

Barnett and Eryilmaz (2014) also show there is only one periodic solution, while the other solutions diverge from the periodic solution as $t \longrightarrow \infty$. This periodic solution is called an unstable limit cycle. The model is not subject to bifurcation within the feasible parameter space, when $\phi_x > 0$ and $\phi_\pi > 0$, although bifurcation is possible within the more general functional structure of system (1.7.7).

ii. Current-Looking Taylor Rule With Interest Rate Smoothing

The model consists of the equations (1.7.4) and (1.7.5), along with the following policy rule having $N = 3$ policy parameters:

$$r_t = \bar{r}_t + \phi_\pi \pi_t + \phi_x x_t + \phi_r r_{t-1}. \tag{1.7.9}$$

The system can be written in the form $E_t \mathbf{y_{t+1}} = \mathbf{C}\mathbf{y_t} + \mathbf{d_t}$ as:

$$\begin{bmatrix} E_t x_{t+1} \\ E_t \pi_{t+1} \\ E_t r_{t+1} \end{bmatrix} = \mathbf{C} \begin{bmatrix} x_t \\ \pi_t \\ r_t \end{bmatrix} + \begin{bmatrix} -\dfrac{1-\alpha+\alpha\omega}{\sigma}\bar{r}_t \\ 0 \\ E_t \bar{r}_{t+1} - \phi_x \bar{r}_t \dfrac{1-\alpha+\alpha\omega}{\sigma} \end{bmatrix} \tag{1.7.10}$$

with

$$\mathbf{y_t} = \begin{bmatrix} x_t \\ \pi_t \\ r_t \end{bmatrix},$$

$\mathbf{C} =$

$$\begin{bmatrix} \frac{\mu}{\beta}\left(1+\varphi\frac{1-\alpha+\alpha\omega}{\sigma}\right)+1 & -\frac{1-\alpha+\alpha\omega}{\sigma\beta} & \frac{1-\alpha+\alpha\omega}{\sigma} \\ -\frac{\mu}{\beta}\left(\frac{\sigma}{1+\alpha(\omega-1)}+\varphi\right) & \frac{1}{\beta} & 0 \\ \phi_x+\frac{\mu}{\beta}\left(1+\varphi\frac{1-\alpha+\alpha\omega}{\sigma}\right)\left(\phi_x\frac{1-\alpha+\alpha\omega}{\sigma}-\phi_\pi\right) & -\frac{1}{\beta}\left(\phi_x\frac{1-\alpha+\alpha\omega}{\sigma}-\phi_\pi\right) & \phi_r+\phi_x\frac{1-\alpha+\alpha\omega}{\sigma} \end{bmatrix}.$$

Assuming the system (1.7.10) has a pair of complex conjugate eigenvalues and a real-valued eigenvalue, the following proposition states the conditions for the system to undergo a Hopf bifurcation.

Proposition 1.7.2 The system (1.7.10) undergoes a Hopf bifurcation at equilibrium points, if and only if the following transversality condition holds

$$\frac{\partial|\lambda_i(\phi)|}{\partial\phi_j}\Big|_{\phi_j=\phi_j^*}\neq 0\,,$$

and also

$$(a)\ \ \phi_r-\beta<0, \tag{1.7.11}$$

$$(b)\ \ \phi_r\left(\frac{\sigma(2+\mu+2\beta)}{1-\alpha+\alpha\omega}+\varphi\mu\right)+\phi_x(\beta+1)+\mu\left(\frac{\sigma}{1-\alpha+\alpha\omega}+\varphi\right)(\phi_\pi+1)$$

$$+\frac{2\sigma}{1+\alpha(\omega-1)}<0, \tag{1.7.12}$$

$$(c)\ \phi_r{}^2\xi_4+\phi_r\xi_3+(\phi_x\phi_r+\phi_x)\xi_2+\phi_\pi\xi_1+\xi_0=-1. \tag{1.7.13}$$

Hopf bifurcation cannot occur in the model, since (1.7.12) does not hold. To detect the existence of a period doubling bifurcation, Barnett and Eryilmaz (2014) keep the structural parameters and policy parameters, ϕ_π and ϕ_r, constant at their baseline values, while varying the policy parameter ϕ_x over a feasible range. They numerically find period doubling bifurcation at $\phi_x = 0.83$. When they consider ϕ_π as the bifurcation parameter, they numerically find a period doubling bifurcation at $\phi_\pi = 5.57$ and a branching point at $\phi_\pi = -1.5$. There is no bifurcation of any type at $(\omega, \alpha) = (0,1)$.

iii. Forward-Looking Taylor Rule

The model consists of equations (1.7.4) and (1.7.5) along with the following policy rule:

$$r_t = \bar{r}_t + \phi_\pi E_t \pi_{t+1} + \phi_x E_t x_{t+1}. \tag{1.7.14}$$

Rearranging terms, the system can be written in the form

$$E_t \mathbf{y_{t+1}} = \mathbf{C}\mathbf{y_t} \, , \tag{1.7.15}$$

with

$$\mathbf{y_t} = \begin{bmatrix} x_t \\ \pi_t \end{bmatrix},$$

$$\mathbf{C} = \begin{bmatrix} \dfrac{\beta\sigma - \left(\mu\sigma + \mu\varphi\left(1+\alpha(\omega-1)\right)\right)(\phi_\pi - 1)}{\beta\sigma - \beta\phi_x\left(1+\alpha(\omega-1)\right)} & \dfrac{(\phi_\pi - 1)\left(1+\alpha(\omega-1)\right)}{\beta\sigma - \beta\phi_x\left(1+\alpha(\omega-1)\right)} \\ -\dfrac{\mu\sigma + \mu\varphi\left(1+\alpha(\omega-1)\right)}{\beta + \alpha\beta(\omega-1)} & \dfrac{1}{\beta} \end{bmatrix}.$$

Assuming the system (1.7.15) has a pair of complex conjugate eigenvalues, the following proposition provides the conditions for the system to undergo a Hopf bifurcatio

Proposition 1.7.3 The system (1.7.15) undergoes a Hopf bifurcation at equilibrium points, if and only if $\Delta < 0$ and

$$\phi_x^* = \frac{\beta - 1}{\beta} \frac{\sigma}{1 + \alpha(\omega - 1)} \tag{1.7.16}$$

Barnett and Eryilmaz (2014) find a period doubling bifurcation at $\phi_x = 1.913$ and a Hopf bifurcation at $\phi_x = -0.01$. Given the baseline values of the parameters, Hopf bifurcation occurs outside the feasible set of parameter values. There is no bifurcation at $(\alpha, \omega) = (1, 0)$. The system has a periodic solution at $\phi_\pi = 2.8$ and $\phi_x = 0$. The origin is a stable spiral point. Any solution that starts around the origin in the phase plane will spiral toward the origin. The origin is a stable sink, since the trajectories spiral inward.

iv. Pure Forward-Looking Inflation Targeting

The model consists of equations (1.7.4) and (1.7.5) along with the following policy rule:

$$r_t = \bar{r}_t + \phi_\pi E_t \pi_{t+1}. \tag{1.7.17}$$

Rearranging the terms, the system can be written in the form

$$E_t \mathbf{y_{t+1}} = \mathbf{C} \mathbf{y_t} \,, \tag{1.7.18}$$

with $\mathbf{y_t} = \begin{bmatrix} x_t \\ \pi_t \end{bmatrix}$,

$$\mathbf{C} = \begin{bmatrix} 1 - \left(\frac{\mu}{\beta} + \frac{\varphi\mu(1+\alpha(\omega-1))}{\beta\sigma}\right)(\phi_\pi - 1) & \frac{(\phi_\pi - 1)(1+\alpha(\omega-1))}{\beta\sigma} \\ -\frac{\mu}{\beta}\left(\frac{\sigma}{1+\alpha(\omega-1)} + \varphi\right) & \frac{1}{\beta} \end{bmatrix}.$$

Assuming the system (1.7.18) has a pair of complex conjugate eigenvalues, the following proposition provides the conditions for the system to undergo a Hopf bifurcation.

Proposition 1.7.4 The system (1.7.18) undergoes a Hopf bifurcation at equilibrium points, if and only if

$$\Delta < 0 \text{ and } \beta^* = 1. \tag{1.7.19}$$

Barnett and Eryilmaz (2014) show that the solution path for $\beta = 1$ and $\phi_\pi = 8$ is periodic and oscillates around the origin, which is a stable center. Hopf bifurcation appears at $\beta = 1$ regardless of the values of α and ω. This result is the same as in the closed economy case under forward-looking inflation targeting in Barnett and Duzhak (2010). But setting the discount factor at 1 is not justifiable for a New Keynesian model, whether within an open or closed economy framework. Barnett and Eryilmaz (2014) also numerically find a period doubling bifurcation at $\beta = -0.91$, which is not theoretically feasible.

Barnett and Eryilmaz (2014) further show that there is only one periodic solution, which is an unstable limit cycle, and other solutions diverge from the periodic solution at $t \longrightarrow \infty$. Varying ϕ_π while setting $\beta = 1$ and keeping the other parameters constant at their baseline values, they numerically find a Hopf bifurcation at $\phi_\pi = 1.0176$, a period doubling bifurcation at $\phi_\pi = 12.76$, and a branching point at $\phi_\pi = 1$.

v. Backward-Looking Taylor Rule

The model consists of equations (1.7.4) and (1.7.5) along with the following policy rule:

$$r_t = \bar{r}_t + \phi_\pi \pi_{t-1} + \phi_x x_{t-1}. \tag{1.7.20}$$

The system can be written in the form $E_t \mathbf{y_{t+1}} = \mathbf{C y_t} + \mathbf{d_t}$:

$$E_t \mathbf{y_{t+1}} = \mathbf{C y_t} + \begin{bmatrix} -\dfrac{1+\alpha(\omega-1)}{\sigma}\bar{r}_t \\ 0 \\ E_t \bar{r}_{t+1} \end{bmatrix}, \tag{1.7.21}$$

with

$$\mathbf{y_t} = \begin{bmatrix} x_t \\ \pi_t \\ r_t \end{bmatrix},$$

$$\mathbf{C} = \begin{bmatrix} \frac{\mu}{\beta}\left(1 + \frac{\varphi(1+\alpha(\omega-1))}{\sigma}\right) + 1 & -\frac{1+\alpha(\omega-1)}{\beta\sigma} & \frac{1+\alpha(\omega-1)}{\sigma} \\ -\frac{\mu}{\beta}\left(\frac{\sigma}{1+\alpha(\omega-1)} + \varphi\right) & \frac{1}{\beta} & 0 \\ \phi_x & \phi_\pi & 0 \end{bmatrix}.$$

Assuming the system (1.7.21) has a pair of complex conjugate eigenvalues, the following proposition provides the conditions for the system to undergo a Hopf bifurcation.

Proposition 1.7.5 The system (1.7.21) undergoes a Hopf bifurcation at equilibrium points, if and only if the transversality condition, $\frac{\partial|\lambda_i(\boldsymbol{\phi})|}{\partial\phi_j}\Big|_{\boldsymbol{\phi}=\boldsymbol{\phi}^*} \neq 0$, holds for some j; and the following conditions also are satisfied:

$$(i)\quad \phi_x + \phi_\pi\mu\left(\frac{\sigma}{1+\alpha(\omega-1)} + \varphi\right) - \frac{\beta\sigma}{1+\alpha(\omega-1)} < 0, \tag{1.7.22}$$

$$(ii)\quad \phi_x(\beta - 1) + \mu\left(\frac{\sigma}{1+\alpha(\omega-1)} + \varphi\right)(1 - \phi_\pi) < 0, \tag{1.7.23}$$

$$(iii)\quad \left(\phi_x + \phi_\pi\mu\left(\frac{\sigma}{1+\alpha(\omega-1)} + \varphi\right)\right)^2 + \left(\phi_x + \phi_\pi\mu\left(\frac{\sigma}{1+\alpha(\omega-1)} + \varphi\right)\right)\xi_1$$

$$-\phi_x\xi_2 = \xi_3. \tag{1.7.24}$$

Barnett and Eryilmaz (2014) numerically find a period doubling bifurcation at $\phi_x = 1.91$. Starting from the point $\phi_x = 1.91$, they construct the period doubling boundary by varying ϕ_x and ϕ_π simultaneously. They also show that along the bifurcation boundary, the positive values of ϕ_x lie between 0 and 13. As the magnitude of ϕ_π increases, smaller values of ϕ_x would be sufficient to cause period doubling bifurcation under a backward-looking policy. Their numerical analysis with CL MatCont detects a codimension-2 fold-flip bifurcation (LPPD) at $(\phi_x, \phi_\pi) = (0.94, 2.01)$ and a flip-Hopf bifurcation (PDNS) at $(\phi_x, \phi_\pi) = (-6.98, 3.36)$. By treating the policy parameter ϕ_π as the potential source of bifurcation, numerical analysis also indicates a period doubling bifurcation at $\phi_\pi = 11.87$. By varying ϕ_π while keeping the other parameters constant at their benchmark values, another period doubling bifurcation is found at relatively large values of the parameter $\phi_\pi = 11.87$, which is nevertheless still within the feasible subset of the parameter space defined by Bullard and Mitra (2002).

vi. Backward-Looking Taylor Rule with Interest Rate Smoothing

The model consists of equations (1.7.4) and (1.7.5) along with the following policy rule:

$$r_t = \bar{r}_t + \phi_\pi \pi_{t-1} + \phi_x x_{t-1} + \phi_\pi r_{t-1}. \tag{1.7.25}$$

The system can be written in the form $E_t \mathbf{y_{t+1}} = \mathbf{C} \mathbf{y_t} + \mathbf{d_t}$:

$$E_t \mathbf{y_{t+1}} = \mathbf{C} \mathbf{y_t} + \begin{bmatrix} -\dfrac{1+\alpha(\omega-1)}{\sigma} \bar{r}_t \\ 0 \\ E_t \bar{r}_{t+1} \end{bmatrix}, \tag{1.7.26}$$

with

$$\mathbf{y_t} = \begin{bmatrix} x_t \\ \pi_t \\ r_t \end{bmatrix},$$

$$\mathbf{C} = \begin{bmatrix} \frac{\mu}{\beta}\left(1+\frac{\varphi(1+\alpha(\omega-1))}{\sigma}\right)+1 & -\frac{1+\alpha(\omega-1)}{\beta\sigma} & \frac{1+\alpha(\omega-1)}{\sigma} \\ -\mu\left(1+\frac{\varphi(1+\alpha(\omega-1))}{\sigma}\right) & \frac{1}{\beta} & 0 \\ \phi_x & \phi_\pi & \phi_r \end{bmatrix}.$$

Proposition 1.7.6. The system (1.7.26) undergoes a Hopf bifurcation at equilibrium points, if and only if the transversality condition, $\frac{\partial|\lambda_i(\mathbf{\Phi})|}{\partial\phi_j}\Big|_{\mathbf{\Phi}=\mathbf{\Phi}^*} \neq 0$, holds for some j; and the following conditions also are satisfied:

$$(i)\ \left|\frac{\phi_x-\phi_r\frac{\sigma}{1+\alpha(\omega-1)}+\phi_\pi(\frac{\sigma\mu}{1+\alpha(\omega-1)}+\varphi\mu)}{\frac{\beta\sigma}{1+\alpha(\omega-1)}}\right|<1,$$

with $\phi_x-\phi_r\xi_2+\phi_\pi\xi_3<\frac{\beta\sigma}{1+\alpha(\omega-1)}$, and $\phi_r<\phi_x\xi_2+\phi_\pi\xi_1+\beta$,

$$(ii)\ \left|\phi_x\frac{1-\alpha+\alpha\omega}{\beta\sigma}-\phi_r\frac{1}{\beta}+\phi_\pi\mu\left(\frac{1}{\beta}+\varphi\frac{1-\alpha+\alpha\omega}{\beta\sigma}\right)-\left(\phi_r+\frac{1+\mu}{\beta}+\varphi\mu\frac{1-\alpha+\alpha\omega}{\beta\sigma}+1\right)\right|$$

$$<1+\phi_r\left(\frac{1+\mu}{\beta}+\varphi\mu\frac{1-\alpha+\alpha\omega}{\beta\sigma}+1\right)-\phi_x\frac{1-\alpha+\alpha\omega}{\beta\sigma}+\frac{1}{\beta}$$

with $\phi_x\xi_2+\phi_\pi\xi_1-(1+\phi_r)\xi_0<0$, and $\phi_x\xi_3-\xi_4(\phi_\pi+\phi_r-1)<0$,

$$\text{(iii)}\quad \phi_r\left(\frac{1+\mu}{\beta}+\varphi\mu\frac{1-\alpha+\alpha\omega}{\beta\sigma}+1\right)-\phi_x\frac{1-\alpha+\alpha\omega}{\sigma}+\frac{1}{\beta}+\Bigg(\phi_x\frac{1-\alpha+\alpha\omega}{\beta\sigma}-\phi_r\frac{1}{\beta}+$$

$$\phi_\pi\mu\left(\frac{1}{\beta}+\varphi\frac{1-\alpha+\alpha\omega}{\beta\sigma}\right)\Bigg)\cdot\left(\phi_r+\frac{1+\mu}{\beta}+\varphi\mu\frac{1-\alpha+\alpha\omega}{\beta\sigma}+1\right)$$

$$=1-\left(\phi_x\frac{1-\alpha+\alpha\omega}{\beta\sigma}-\phi_r\frac{1}{\beta}+\phi_\pi\mu\left(\frac{1}{\beta}+\varphi\frac{1-\alpha+\alpha\omega}{\beta\sigma}\right)\right)^2.$$

Barnett and Eryilmaz (2014) detect a period doubling bifurcation numerically at $\phi_x = 3$, given the benchmark values of the parameters and the setting $\phi_r = 0.5$. When

$\phi_r = 1$, period doubling bifurcation occurs at $\phi_x = 4.09$. They find bifurcation boundary by varying ϕ_x and ϕ_π simultaneously, and then ϕ_x and ϕ_r simultaneously. In (ϕ_x, ϕ_π)-space, the bifurcation boundary lies within the narrow range from $\phi_x = 3$ and $\phi_x = 3.25$. In contrast, ϕ_x varies more elastically in response to changes in ϕ_r along the bifurcation boundary in (ϕ_r, ϕ_x)-space.

Barnett and Eryilmaz (2014) further find codimension-2 fold-flip bifurcations at $(\phi_x, \phi_\pi) = (0.41{,}3.19)$ and at $(\phi_x, \phi_r) = (0.78, -0.52)$, as well as flip-Hopf bifurcations at $(\phi_x, \phi_\pi) = (-10.44, 5.04)$ and $(\phi_x, \phi_r) = (-0.74, -1.23)$. Bifurcation disappears at $(\alpha, \omega) = (1{,}0)$.

vii. Hybrid Taylor Rule

The model consists of equations (1.7.4) and (1.7.5) along with the following policy rule:

$$r_t = \bar{r}_t + \phi_\pi E_t \pi_{t+1} + \phi_x x_t \tag{1.7.27}$$

The system can be written in the form:

$$E_t \mathbf{y_{t+1}} = \mathbf{C}\mathbf{y_t}\,, \tag{1.7.28}$$

with

$$\mathbf{y_t} = \begin{bmatrix} x_t \\ \pi_t \end{bmatrix},$$

$$
\mathbf{C} = \begin{bmatrix} \dfrac{\beta\phi_x + \mu(\dfrac{\sigma}{1+\alpha(\omega-1)} + \varphi)(1-\phi_\pi)}{\dfrac{\beta\sigma}{1+\alpha(\omega-1)}} + 1 & \dfrac{(\phi_\pi - 1)\big(1+\alpha(\omega-1)\big)}{\beta\sigma} \\ -\dfrac{\mu}{\beta}(\dfrac{\sigma}{1+\alpha(\omega-1)} + \varphi) & \dfrac{1}{\beta} \end{bmatrix}.
$$

Proposition 1.7.7. The system (1.7.28) exhibits a Hopf bifurcation at equilibrium points, if and only if $\Delta < 0$ and

$$
\phi_x^* = \frac{\sigma(\beta-1)}{1+\alpha(\omega-1)}. \qquad (1.7.29)
$$

Barnett and Eryilmaz (2014) find a period doubling bifurcation at $\phi_x = -1.92$ as well as a Hopf bifurcation at $\phi_x = -0.01$, while system parameters are at benchmark values. Assuming positive values for policy parameters, values of the bifurcation parameters are outside the feasible region of the parameter space. They conclude that the feasible set of parameter values for ϕ_x does not include a bifurcation boundary. They also find that in the(ϕ_π, ϕ_x)-space, along the period-doubling bifurcation boundary, the bifurcation parameter ϕ_x varies in the same direction as ϕ_π. Therefore as ϕ_x increases, higher values of ϕ_x are required to cause a period doubling bifurcation. They analyze the solution paths from (1.7.28) with stability properties indicating Hopf bifurcation. The inner spiral trajectory is converging to the equilibrium point, while the outer spiral is diverging.

1.7.3 Clarida, Gali, and Gertler Model

Barnett and Eryilmaz (2013) conduct bifurcation analysis in the open-economy New Keynesian model developed by Clarida, Gali, and Gertler (2002). Clarida, Gali, and Gertler (2002) developed a two-country version of a small open economy model, based on Clarida, Gali, and Gertler (2001) and Gali and Monacelli (1999).

Following Walsh (2003, pp.539-540), the model of Clarida, Gali, and Gertler (2002) can be written as follows:

$$\pi_t^h = \beta E_t \pi_{t+1}^h + \delta \left[\sigma + \eta + \left(\frac{v\sigma}{1+w}\right)\right] x_t, \tag{1.7.30}$$

$$x_t = E_t x_{t+1} - \left(\frac{1+w}{\sigma}\right)\left(r_t - E_t \pi_{t+1}^h - \bar{r}_t\right), \tag{1.7.31}$$

$$r_t = \bar{r}_t + \phi_\pi \pi_t^h + \phi_x x_t. \tag{1.7.32}$$

Equation (1.7.30) is an inflation adjustment equation for the aggregate price of domestically produced goods. Equation (1.7.31) is the dynamic IS curve, derived from the Euler condition of the consumers' optimization problem. The monetary policy rule, (1.7.32), is a domestic-inflation-based current-looking Taylor rule.

Let x_t denote the output gap, π_t^h the inflation rate for domestically produced goods and services, and r_t the nominal interest rate, with E_t being the expectation operator and $\bar{r}_t$ denoting the small open economy's natural rate of interest. The lowercase letter denotes the logs of the respective variables. The coefficients $\phi_x > 0$ and $\phi_\pi > 0$ are the policy parameters, measuring the sensitivity of the nominal interest rate to changes in output gap and inflation rate, respectively. In addition, $\delta = [(1-\theta)(1-\beta\theta)]/\theta$ is a composite parameter with θ representing the probability that a firm holds its price unchanged in a given period of time, while $1-\theta$ is the probability that a firm resets its price. The parameter η denotes the wage elasticity of labor demand, and σ^{-1} denotes the elasticity of intertemporal substitution. The parameter w denotes the growth rate of nominal wages, $\rho = \beta^{-1} - 1$ is the time discount rate, and v is the population size in the foreign country, with $1 - v$ being the population size of the home country. Wealth effect is captured by the term $v\sigma$.[19]

Substituting (1.7.32) for $r_t - \bar{r}_t$ into the equation (1.7.31), Barnett and Eryilmaz (2013) reduce the system to a first order dynamical system in two equations for domestic inflation and output gap. The system is given by:

$$\pi_t^h = \beta E_t \pi_{t+1}^h + \delta \left[\sigma + \eta + \left(\frac{v\sigma}{1+w} \right) \right] x_t,$$

$$x_t = E_t x_{t+1} - \left(\frac{1+w}{\sigma} \right) \left(\phi_\pi \pi_t^h + \phi_x x_t - E_t \pi_{t+1}^h \right).$$

An equilibrium solution to the system is $x_t = \pi_t^h = 0$ for all t. The system can be written in the standard form as

$$\mathbf{A} E_t \mathbf{y_{t+1}} = \mathbf{B} \mathbf{y_t}, \tag{1.7.33}$$

or $E_t \mathbf{y_{t+1}} = \mathbf{C} \mathbf{y_t}$, where $\mathbf{C} = \mathbf{A^{-1}} \mathbf{B}$, as follows:

$$\begin{bmatrix} E_t x_{t+1} \\ E_t \pi_{t+1}^h \end{bmatrix} = \mathbf{C} \begin{bmatrix} x_t \\ \pi_t^h \end{bmatrix}, \tag{1.7.34}$$

where

$$\mathbf{C} = \begin{bmatrix} 1 + \frac{(1+w)\phi_x}{\sigma} + \delta(1+w) \left(\sigma + \eta + \left(\frac{v\sigma}{1+w} \right) \right) \frac{1}{\beta\sigma} & \frac{(1+w)\phi_\pi}{\sigma} - \frac{(1+w)}{\beta\sigma} \\ -\delta(\sigma + \eta + \left(\frac{v\sigma}{1+w} \right)) \frac{1}{\beta} & \frac{1}{\beta} \end{bmatrix}.$$

Assuming a pair of complex conjugate eigenvalues, the conditions for the existence of a Hopf bifurcation are provided in the following proposition.

[19] The model description is modified from Barnett and Eryilmaz (2013).

Proposition 1.7.8. Let Δ be the discriminant of the characteristic equations. Then the system (1.7.34) undergoes a Hopf bifurcation at equilibrium points, if and only if $\Delta < 0$ and

$$\phi_x^* = \frac{\beta\sigma-1}{1+w} - \phi_\pi\left(\frac{\delta\sigma(1+v+w)}{1+w} + \delta\eta\right). \tag{1.7.35}$$

Proof. See Barnett and Eryilmaz (2013), Proposition 1.

Barnett and Eryilmaz (2013) observe that the Clarida, Gali, and Gertler (2002) model differs in several aspects from the Gali and Monacelli (2005) model. The degree to which the two models differ depends upon the parameter settings. In the Clarida, Gali, and Gertler (2002) model, the parameters w, v, and δ play an important role in determining the critical value of the bifurcation parameter. Barnett and Eryilmaz (2013) note that numerical implementation of the theory to locating Hopf bifurcation boundaries in the Clarida, Gali, and Gerler (2002) model would be a challenging project.

1.8 Two Endogenous Growth Models[20]

1.8.1 Introduction

This section surveys Barnett and Ghosh (2013, 2014) about bifurcation analyses of two endogenous growth models. Previous stability analyses of endogenous growth models include the following. Benhabib and Perli (1994) analyzed the stability property of the long-run equilibrium in the Lucas (1988) model; Arnold (2000a, 2000b) analyzed the stability of equilibrium in the Romer (1990) model; Arnold (2006) has done the same for the Jones (1995) model; and Mondal (2008) examined the dynamics of the Grossman-Helpman (1991) model of endogenous product cycles. The results derived in those papers provide important

[20] This section is summarized from Barnett and Ghosh (2013, 2014).

insights to researchers. But a detailed bifurcation analysis had not been provided for many of these popular endogenous growth models. Barnett and Ghosh (2014) filled the gap for the Uzawa-Lucas endogenous growth model, as surveyed in section 8.2 below, while Barnett and Ghosh (2013) do so for a variant of Jones (2002) semi-endogenous growth model, as surveyed in section 1.8.3.

In section 1.8.2, Barnett and Ghosh (2014) conduct bifurcation analysis on the Uzawa-Lucas endogenous growth model, which is solved from a centralized social planner perspective as well as in the model's decentralized market economy form. Barnett and Ghosh (2014) locate transcritical bifurcation and Hopf bifurcation boundaries for the decentralized version of the model using Mathematica, and also investigate the existence of Hopf bifurcation, branch point bifurcation, limit point cycle bifurcation, and period doubling bifurcations using Matcont. The series of period doubling bifurcations confirm the existence of global bifurcation and reveal the possibility of chaotic dynamics. Barnett and Ghosh (2014) also point out that the externality of the human capital parameter plays an important role in determining the dynamics of the decentralized model. On the contrary, from the centralized social planner perspective, the solution is saddle path stable with no possibility of bifurcation within the feasible parameter set.

In section 1.8.3, Barnett and Ghosh (2013) conduct bifurcation analysis on a variant of the Jones (2002) model. Jones found that long-run growth arises from the worldwide discovery of ideas, which depend on the rate of population growth of the countries contributing to world research rather than on the level of population. His model exhibits "weak" scale effect, in contrast with the "strong" scale effect, produced by the first generation endogenous growth models of Romer (1990) and Grossman and Helpman (1991).

Barnett and Ghosh (2013) incorporate human capital accumulation into a Jones model. They also consider the possibility that the direction of technology progress is driven by human capital investment (Bucci (2008)). As a result, the parameters in the human capital accumulation equation play an important role in determining the dynamics of the model. Barnett and Ghosh (2013) also introduce the possibility of decreasing returns to scale associated with human capital and with time spent accumulating human capital in the production equation. This assumption accounts for the scale effects in the model and permits a closed form solutions for the steady state of the model. Using the numerical package Matcont, Barnett and Ghosh (2013) further show the existence of Hopf bifurcation, branch point bifurcation, limit point of cycles, Bogdanov-Takens bifurcation, and generalized Hopf bifurcations within the feasible parameter sets.

In both models, Barnett and Ghosh (2013, 2014) emphasize that bifurcation boundaries do not necessarily separate stable from unstable solution domains. Barnett and Ghosh (2013, 2014) note that bifurcation boundaries can separate one kind of unstable dynamics domain from another kind of unstable dynamics domain. Not as well known is that bifurcation boundaries can separate one kind of stable dynamics domain from another kind of stable dynamics domain (called soft bifurcation), such as bifurcation from monotonic stability to damped periodic stability or from damped periodic to damped multiperiodic stability. Recognizing there are an infinite number of kinds of unstable dynamics as well as an infinite number of kinds of stable dynamics, subjective prior views on the stability of economies are not reliable without conducting analysis of model dynamics.

1.8.2 Uzawa-Lucas Endogenous Growth Model[21]

The Uzawa-Lucas endogenous growth model (Uzawa (1965) and Lucas (1988)) is one of the most important endogenous growth models. This model has two sectors: the human capital production sector and the physical capital production sector, producing human capital and physical capital, respectively. Individuals have the same level of work qualification and expertise (H). They allocate some of their time to producing final goods and dedicate the remaining time to training and studying. Barnett and Ghosh (2014) solve the model from a centralized social planner perspective as well as from the model's decentralized market economy form.

The production function in the physical sector is defined as follows:

$$Y = AK^{\alpha}(\varepsilon hL)^{1-\alpha}h_a^{\zeta}, \qquad 0 < \alpha < 1,$$

where Y is output, A is technology level, K is physical capital, α is the share of physical capital, L is labor, and h is human capital per person. In addition, ε and $1-\varepsilon$ are respectively the fraction of labor time devoted to producing output and human capital, where $0 < \varepsilon < 1$. Observe that εhL is the quantity of labor, measured in efficiency units, employed to produce output, and h_a^{ζ} measures the externality associated with average human capital of the work force h_a, where ζ is the positive externality parameter in the production of human capital. In per capita terms, $y = Ak^{\alpha}(\varepsilon h)^{1-\alpha}h_a^{\zeta}$.

The physical capital accumulation equation is

$$\dot{K} = AK^{\alpha}(\varepsilon hL)^{1-\alpha}h_a^{\zeta} - C - \delta K.$$

[21] The model description is modified from Barnett and Ghosh (2014).

In per capita terms, the equation is

$$\dot{k} = Ak^{\alpha}(\varepsilon h)^{1-\alpha}h_a^{\zeta} - c - (n+\delta)k,$$

and the human capital accumulation equation is

$$\dot{h} = \eta h(1-\varepsilon),$$

where η is defined as schooling productivity.

The decision problem is

$$\max_{c_t,\, \varepsilon_t} \int_t^{\infty} \frac{e^{-(\rho-n)t}\left(c(\tau)^{1-\sigma}-1\right)}{1-\sigma}dt\,, \tag{1.8.1}$$

subject to

$$\dot{k} = Ak^{\alpha}(\varepsilon h)^{1-\alpha}h_a^{\zeta} - c - (n+\delta)k \tag{1.8.2}$$

and

$$\dot{h} = \eta(1-\varepsilon)h, \tag{1.8.3}$$

where ρ ($\rho > n > 0$) is the subjective discount rate, and $\sigma \geq 0$ is the inverse of the intertemporal elasticity of substitution in consumption.

i. Social Planner Problem

The social planner takes into account the externality associated with human capital, when solving the maximization problem (1.8.1) subject to (1.8.2) and (1.8.3). From the first order conditions, Barnett and Ghosh (2014, Appendix 2) derive the equations describing the economy of the Uzawa-Lucas model from a social planner's perspective:

$$\frac{\dot{k}}{k} = Ak^{\alpha-1}\varepsilon^{1-\alpha}h^{1-\alpha+\zeta} - \frac{c}{k} - (n+\delta),$$

$$\frac{\dot{h}}{h} = \eta(1-\varepsilon),$$

$$\frac{\dot{c}}{c} = \frac{\alpha Ak^{\alpha-1}\varepsilon^{1-\alpha}h^{1-\alpha+\zeta} - (\rho+\delta)}{\sigma},$$

$$\frac{\dot{\varepsilon}}{\varepsilon} = \eta\frac{(1-\alpha+\zeta)}{1-\alpha}\varepsilon + \eta\frac{(1-\alpha+\zeta)}{\alpha} - \frac{c}{k} + \frac{(1-\alpha)}{\alpha}(n+\delta),$$

$$\frac{\dot{L}}{L} = n.$$

Let $m = \dfrac{Y}{K}$ and $g = \dfrac{c}{k}$. Taking logarithms of m and g and differentiating with respect to time, the dynamics of the Uzawa-Lucas model is given by equation (1.8.4) and (1.8.5):

$$\frac{\dot{m}}{m} = -(1-\alpha)m + \frac{1-\alpha}{\alpha}(n+\delta) + \eta\frac{(1-\alpha+\zeta)}{\alpha}. \tag{1.8.4}$$

$$\frac{\dot{g}}{g} = \left(\frac{\alpha}{\sigma} - 1\right)m - \frac{\rho}{\sigma} - \delta\left(\frac{1}{\sigma} - 1\right) + g + n. \tag{1.8.5}$$

The steady state (m^*, g^*) is given by $\dot{m} = \dot{g} = 0$ and is derived to be

$$m^* = \eta\frac{(1-\alpha+\zeta)}{\alpha} + \frac{(n+\delta)}{\alpha},$$

$$g^* = \frac{\rho-n}{\sigma} + \frac{1-\alpha}{\alpha}(n+\delta) + \eta\frac{(1-\alpha+\zeta)}{\alpha(1-\alpha)}\frac{(\sigma-\alpha)}{\sigma}.$$

A unique steady state exists, if

$$\Lambda = \frac{(1-\alpha+\zeta)}{\alpha}(\sigma-1)\eta(1-\varepsilon)+\rho > 0.$$

This inequality condition for Λ is the transversality condition for the consumer's utility maximization problem, as shown in Barnett and Ghosh (2014, Appendix 1). It can be shown that the social planner solution is saddle path stable. See, e.g., Barro and Sala-i-Martín (2003) and Mattana (2004). Linearizing around the steady state, $s^* = (m^*, g^*)$, the local stability properties of the system defined by equations (1.8.4) and (1.8.5) can be found. The result is

$$\begin{bmatrix} \dot{m} \\ \dot{g} \end{bmatrix} = \underbrace{\begin{bmatrix} \frac{\partial \dot{m}}{\partial m}\Big|_{s^*} & \frac{\partial \dot{m}}{\partial g}\Big|_{s^*} \\ \frac{\partial \dot{g}}{\partial m}\Big|_{s^*} & \frac{\partial \dot{g}}{\partial g}\Big|_{s^*} \end{bmatrix}}_{\mathbf{J_s}} \begin{bmatrix} m_t - m^* \\ g_t - g^* \end{bmatrix},$$

where

$$\mathbf{J_s} = \begin{bmatrix} -(1-\alpha)m^* & 0 \\ \left(\frac{\alpha}{\sigma}-1\right)g^* & g^* \end{bmatrix}.$$

Since $m^* > 0$ and $g^* > 0$, it follows that $det(\mathbf{J_s}) = -(1-\alpha)m^* g^* < 0$. Hence the saddle path is stable.

ii. Representative Agent Problem

From the first order conditions with $h = h_a$, Barnett and Ghosh (2014, Appendix 3) derive the following equations describing the dynamics of the decentralized Uzawa-Lucas model:

$$\frac{\dot{k}}{k} = Ak^{\alpha-1}\varepsilon^{1-\alpha}h^{1-\alpha+\zeta} - \frac{c}{k} - (n+\delta),$$

$$\frac{\dot{h}}{h} = \eta(1-\varepsilon),$$

$$\frac{\dot{c}}{c} = \frac{\alpha Ak^{\alpha-1}\varepsilon^{1-\alpha}h^{1-\alpha+\zeta} - (\rho+\delta)}{\sigma},$$

$$\frac{\dot{\varepsilon}}{\varepsilon} = \eta\frac{(\alpha-\zeta)}{1-\alpha}\varepsilon + \eta\frac{(1-\alpha+\zeta)}{\alpha} - \frac{c}{k} + \frac{(1-\alpha)}{\alpha}(n+\delta),$$

$$\frac{\dot{L}}{L} = n.$$

Taking logarithms of m and g and differentiating with respect to time, the following three equations define the dynamics of the Uzawa-Lucas model

$$\frac{\dot{m}}{m} = -(1-\alpha)m + \frac{(1-\alpha)}{\alpha}(n+\delta) + \eta\frac{(1-\alpha+\zeta)}{\alpha} - \eta\frac{\zeta}{\alpha}\varepsilon\,, \tag{1.8.6}$$

$$\frac{\dot{g}}{g} = \left(\frac{\alpha}{\sigma}-1\right)m - \frac{\rho}{\sigma} - \delta\left(\frac{1}{\sigma}-1\right) + g + n, \tag{1.8.7}$$

$$\frac{\dot{\varepsilon}}{\varepsilon} = \eta\frac{(\alpha-\zeta)}{\alpha}\varepsilon + \eta\frac{(1-\alpha+\zeta)}{\alpha} - g + \frac{(1-\alpha)}{\alpha}(n+\delta). \tag{1.8.8}$$

The steady state $(m^*, g^*, \varepsilon^*)$, given by $\dot{m} = \dot{g} = \dot{\varepsilon} = 0$, is

$$\varepsilon^* = 1 - \frac{(1-\alpha)(\rho-n-\eta)}{\eta[\zeta-\sigma(1-\alpha+\zeta)]},$$

$$m^* = \eta\frac{[1-\alpha+\zeta(1-\varepsilon^*)]}{\alpha(1-\alpha)} + \frac{n}{\alpha},$$

$$g^* = \eta\frac{[1-\alpha+\zeta(1-\varepsilon^*)+\alpha\varepsilon^*]}{\alpha(1-\alpha)} + \frac{n(1-\alpha)}{\alpha}.$$

A unique steady state exists, if

$$\Lambda = \frac{(1-\alpha+\zeta)}{\alpha}(\sigma-1)\eta(1-\varepsilon)+\rho > 0,$$

and $0 < \varepsilon < 1$.

The inequality condition on Λ is the transversality condition for the consumer's utility maximization problem (Barnett and Ghosh (2014), appendix 1), while $0 < \varepsilon^* < 1$ is necessary for $m^*, g^* > 0$. Linearizing the system around the steady state, $s^* = (m^*, g^*, \varepsilon^*)$, yields the following:

$$\begin{bmatrix} \dot{m} \\ \dot{g} \\ \dot{\varepsilon} \end{bmatrix} = \underbrace{\begin{bmatrix} \frac{\partial \dot{m}}{\partial m}|_{s^*} & \frac{\partial \dot{m}}{\partial g}|_{s^*} & \frac{\partial \dot{m}}{\partial \varepsilon}|_{s^*} \\ \frac{\partial \dot{g}}{\partial m}|_{s^*} & \frac{\partial \dot{g}}{\partial g}|_{s^*} & \frac{\partial \dot{g}}{\partial \varepsilon}|_{s^*} \\ \frac{\partial \dot{\varepsilon}}{\partial m}|_{s^*} & \frac{\partial \dot{\varepsilon}}{\partial g}|_{s^*} & \frac{\partial \dot{\varepsilon}}{\partial \varepsilon}|_{s^*} \end{bmatrix}}_{\mathbf{J_m}} \begin{bmatrix} m_t - m^* \\ g_t - g^* \\ \varepsilon_t - \varepsilon^* \end{bmatrix},$$

where

$$\mathbf{J_m} = \begin{bmatrix} -(1-\alpha)m^* & 0 & -\eta\frac{\zeta}{\alpha}m^* \\ \left(\frac{\alpha}{\sigma}-1\right)g^* & g^* & 0 \\ 0 & -\varepsilon^* & \eta\frac{(\alpha-\zeta)}{\alpha}\varepsilon^* \end{bmatrix}.$$

The characteristic equation associated with $\mathbf{J_m}$ is $q^3 + c_2 q^2 + c_1 q + c_0 = 0$, where

$$c_0 = \eta\frac{[\sigma(1-\alpha+\zeta)-\zeta]}{\sigma}m^* g^* \varepsilon^*,$$

$$c_1 = \eta^2\frac{(\alpha-\zeta)}{\alpha}\varepsilon^{*2} - (1-\alpha)m^* g^*,$$

$$c_2 = -\eta \frac{(2\alpha - \zeta)}{\alpha} \varepsilon^*.$$

iii. Bifurcation Analysis

Barnett and Ghosh (2014) analyze the existence of codimension 1 and 2, transcritical, and Hopf bifurcation in the system ((1.8.6), (1.8.7), (1.8.8)). They search for the bifurcation boundary according to $c_0 = \det(\mathbf{J_m}) = 0$.

Theorem 1.8.1 $\mathbf{J_m}$ has zero eigenvalues, if

$$\eta \frac{[\sigma(1-\alpha+\zeta)-\zeta]}{\sigma} m^* g^* \varepsilon^* = 0. \tag{1.8.9}$$

It follows from the Hopf Bifurcation Thereom in Guckenheimer and Holmes (1983), that if $c_0 - c_1 c_2 = 0$ and $c_1 > 0$, then $\mathbf{J_m}$ has precisely one pair of purely imaginary eigenvalues. But if $c_0 - c_1 c_2 \neq 0$ and $c_1 > 0$, then $\mathbf{J_m}$ has no purely imaginary eigenvalues. Therefore, Barnett and Ghosh (2014) derive the following theorem:

Theorem 1.8.2 The matrix $\mathbf{J_m}$ has precisely one pair of pure imaginary eigenvalues, if

$$\begin{cases} \alpha m^* g^* \big((\alpha - 1)\alpha\sigma + \zeta(\sigma - \alpha)\big) + \eta^2 \sigma \varepsilon^{*2} (2\alpha - \zeta)(\alpha - \zeta) = 0, \\ \qquad \qquad and \\ \frac{\eta^2}{\alpha} \varepsilon^{*2}(\alpha - \zeta) - (1 - \alpha) m^* g^* > 0. \end{cases} \tag{1.8.10}$$

Furthermore, Barnett and Ghosh (2014) explain cyclical behavior in the model. They state that the increase of ζ would bring about the increase of savings rate since consumers are willing to cut current consumption in exchange for higher future consumptions. Then the movement of labor from output production to human capital production brings an increase in human capital, and subsequently faster accumulation of physical capital, if sufficient externality to human capital in production of physical capital is present. On the other hand, a lower subjective discount rate, ρ, could cause consumption to rise gradually with faster capital accumulation. This leads to greater consumption-goods production in the future, which eventually leads to a decline in savings rate. A cyclical convergence to equilibrium comes from these two opposing effects, when savings rate is different from the equilibrium rate. Barnett and Ghosh (2014) conclude that interaction between different parameters can cause cyclical convergence to equilibrium or may cause instability, and for some parameter values convergence to cycles may occur.

Based on Benhabib and Perli (1994), Barnett and Ghosh (2014) locate bifurcation boundaries by keeping some parameters free, while setting the others fixed at $\boldsymbol{\vartheta}^* = \{\eta, \zeta, \alpha, \rho, \sigma, n, \delta\} = (0.05, 0.1, 0.65, 0.0505, 0.15, 0, 0)$ or $\boldsymbol{\omega}^* = \{\eta, \zeta, \alpha, \rho, \sigma, n, \delta\} = (0.05, 0.1, 0.75, 0.0505, 0.15, 0, 0)$. Using Matcont, Barnett and Ghosh (2014) then investigate the stability properties of cycles generated by different combinations of parameters. Some limit cycles, such as supercritical bifurcations, are stable, while some other limit cycles, such as subcritical bifurcations, are unstable. A positive value of the first Lyapunov coefficient indicates creation of subcritical Hopf bifurcation. Period doubling bifurcation occurs, when a new limit cycle, the period of which is twice that of the old one, emerges from an existing limit cycle.

Table 1.8.1 reports the values of the share of capital, α, the externality in production of human capital, ζ, and the inverse of the intertemporal elasticity of substitution in consumption, σ[22]. Since each of the cases reported in Table 1.8.1 has positive first Lyapunov coefficient, an unstable limit cycle (i.e., periodic orbit) bifurcates from the equilibrium.

[22] Table 1.8.1 is a replicate of Barnett and Ghosh (2014) Table 1.

When α is the free parameter, Barnett and Ghosh (2014) find from continuing computation of limit cycles from the Hopf point, that two limit cycles with different periods are present near the limit point cycle (LPC) point at $\alpha = 0.738$. Continuing computation further, a series of period doubling (flip) bifurcations arise. The first period doubling bifurcation at $\alpha = 0.7132369$ has positive normal form coefficients, while the other period doubling bifurcations have negative normal form coefficients. This indicates that the first period doubling bifurcation has unstable double-period cycles, while the rest have stable double-period cycles. Barnett and Ghosh (2014) also find that the limit cycle approaches a global homoclinic orbit, which is a dynamical system trajectory joining a saddle equilibrium point to itself. They also point out the possibility of reaching chaotic dynamics through a series of period doubling bifurcation.

When ζ and σ are free parameters, Barnett and Ghosh (2014) conduct the bifurcation analysis in a similar way by carrying out the continuation of the limit cycle from the first Hopf point. They find that both cases give rise to the LPC point with a nonzero normal form coefficient, indicating the existence of a fold bifurcation at the LPC point.

Table 1.8.1 Stability Analysis Of Uzawa-Lucas Growth Model

Parameters	Equilibrium Bifurcation	Bifurcation of Limit Cycle
α Other parameters set at ϑ^*	**Hopf (H)** First Lyapunov coefficient = 0.00242, α=0.738207	**Limit Point Cycle (LPC)** period= 231.206, α=0.7382042,normal form coefficient=0.007 **Period Doubling (PD)** period= 584.064, α=0.7132369,normal form coefficient=0.910 **Period Doubling (PD)** period= 664.005, α=0.7132002,normal form coefficient=-0.576 **Period Doubling (PD)** period= 693.988, α=0.7131958,normal form coefficient=-0.469 **Period Doubling (PD)** period= 713.978, α=0.7131940,normal form coefficient=-0.368 **Period Doubling (PD)** period= 725.667, α=0.7131932,normal form coefficient=-0.314 **Period Doubling (PD)** period= 784.104, α=0.7131912,normal form coefficient=-0.119

ζ Other parameters set at ω^*	**Hopf (H)** First Lyapunov coefficient =0.00250,ζ=0.107315 **Hopf (H)** First Lyapunov coefficient =0.00246,ζ=0.047059 **Branch Point (BP)** ζ=0.047059	**Limit Point Cycle (LPC)** period= 215.751, ζ=0.1073147,normal form coefficient=0.009
σ Other parameters set at ω^*	**Hopf (H)** First Lyapunov coefficient =0.00264, σ=0.278571 **Hopf (H)** First Lyapunov coefficient =0.00249, σ=0.13939 **Branch Point (BP)** σ=0.278571	**Limit Point Cycle (LPC)** Period= 213.83, σ=0.1394026,normal form coefficient=0.009

1.8.3 Jones Semi-Endogenous Growth Model[23]

The model is based on a variant of Jones'(2002) semi-endogenous growth model.

The labor endowment equation is given by

$$L_{A_t} + L_{Y_t} = L_t = \varepsilon_t N_t, \qquad (1.8.11)$$

where at time t, L_t is employment, L_{Y_t} is the labor employed in producing output, L_{A_t} is the total number of researchers, and N_t is the total population having rate of growth $n > 0$. Each person is endowed with one unit of time and divides the time among producing goods,

[23] The model description is modified from Barnett and Ghosh (2013)

producing ideas and human capital, while ε_t and $1-\varepsilon_t$ represent respectively the amount of time the person spends producing output and accumulating human capital.

The capital accumulation equation is given by

$$\dot{K} = s_{k_t} Y_t - dK_t, \qquad K_0 > 0, \tag{1.8.12}$$

and $$\dot{K} = Y_t - C_t - dK_t, \tag{1.8.13}$$

where s_{k_t} is the fraction of output invested, d is the exogenous, constant rate of depreciation, Y_t is the aggregate production of homogenous final goods, and K_t is capital stock.

Output is produced using the total quantity of human capital, H_{Y_t}, and a set of intermediate goods. The total quantity of human capital equation is given by

$$H_{Y_t} = h_t L_{Y_t}, \tag{1.8.14}$$

with the individual's human capital accumulation equation is given by

$$\dot{h}_t = \eta {h_t}^{\beta_1}(1-\varepsilon_t)^{\beta_2} - \theta g_A h_t, \quad 0 < \beta_1, \beta_2, \varepsilon_t < 1, \eta > 0, 1+\theta > 0, \tag{1.8.15}$$

where h_t is human capital per person and L_{Y_t} is labor employed in producing output. The parameter η is productivity of human capital in the production of new human capital, θ reflects the effect of technological progress on human capital investment, and $g_A = \frac{\dot{A}}{A}$ is the growth rate of technology. Equation (1.8.15) builds on the human capital accumulation equation from the Uzawa-Lucas model.

As noted in Barnett and Ghosh (2013), the human capital accumulation equation has two advantages. It accounts for the scale effects present in the model, and it makes the model tractable to solve for possible steady states. To see this, Barnett and Ghosh (2013) introduced the assumption of decreasing returns to scale of the human capital growth rate in (1.8.15) by setting $0 < \beta_1$ and $\beta_2 < 1$. The higher the level of human capital or of time spent accumulating human capital, the more difficult it is to generate additional human capital. If β_1 or β_2 is equal to 1, the model will exhibit “strong” scale effects. In models associated with strong scale effects, the growth rate of the economy is an increasing function of the population. But this phenomenon is inconsistent with United States data, as shown by Jones (1995). Barnett and Ghosh (2013) also include the technological growth rate, g_A, which directly influences the human capital growth rate. As in Bucci (2008), Barnett and Ghosh (2013) restrict $\theta > -1$ to prevent explosive or negative long run growth rates.

In Barnett and Ghosh (2013), the production function is given by

$$Y_t = {H_{Y_t}}^{1-\alpha} \int_0^A x(i)^\alpha \, di, \tag{1.8.16}$$

where $x(i)$ is the input of intermediate good i, A is the number of available intermediate goods, and $\alpha \in (0,1)$, where $\frac{1}{1-\alpha}$ is the elasticity of substitution for any pair of intermediate goods.

Since research and development (R&D) enable firms to produce new intermediate goods, the R&D technology equation is given by

$$\dot{A} = \gamma {H_{A_t}}^{\lambda} {A_t}^{1-\phi}, \qquad \phi > 0, 0 < \lambda \le 1. \tag{1.8.17}$$

with

$$H_{A_t} = h_t L_{A_t}, \tag{1.8.18}$$

where H_{A_t} is effective research effort and A_t is the existing stock of ideas, while ϕ represents the externalities associated with R&D.

In the final goods sector, the representative final output firm rents capital goods, $x(i)$, from monopolist i at price $p(i)$ and pays w as the rental rate per unit of human capital employed. For each durable, the firm chooses quantity $x(i)$ and H_y to maximize the profit as follows:

$$\max_{x,H_y} \int_0^\infty [H_y^{1-\alpha} x(i)^\alpha - p(i)x(i)]\, di - wH_y.$$

Solving the maximization problem gives

$$p(i) = \alpha H_y^{1-\alpha} x(i)^{\alpha-1}, \tag{1.8.19}$$

$$w = (1-\alpha)\frac{Y}{H_y}. \tag{1.8.20}$$

In the intermediate goods sector, each intermediate good, $x(i)$, is produced by a monopolist, who owns an infinitely-lived patent on a technology determining how to transform a unit of raw material, K, costlessly into intermediate goods. That production function is simply $x = K$. The producer of each specialized durable takes $p(i)$ as given from equation (1.8.19) in choosing the profit maximizing output, x, according to the profit level

$$\pi = \max_x p(x)x - rx,$$

where r is the rental price of raw capital. Solving the monopoly profit maximization problem gives

$$p(i) = \bar{p} = \frac{r}{\alpha}. \tag{1.8.21}$$

The flow of monopoly profit is

$$\pi(i) = \bar{\pi} = \bar{p}\bar{x} - r\bar{x} = (1-\alpha)\bar{p}\bar{x}. \tag{1.8.22}$$

In the research and development sector, the decision to produce a new specialized input depends on a comparison of the discounted stream of net revenue and the cost of the initial investment in a design. Because the market for designs is competitive, the price for designs, P_A, will be bid up until equal to the present value of the net revenue that a monopolist can extract. Therefore P_A is equal to

$$\int_t^{\infty} e^{-\int_t^{\tau} r(s)\, ds} \pi(\tau)\, d\tau = P_A(t), \tag{1.8.23}$$

where r is the interest rate.

If $v(t)$ denotes the value of the innovation, then

$$v(t) = \int_t^{\infty} e^{-\int_t^{\tau} r(s)\, ds} \pi(\tau)\, d\tau. \tag{1.8.24}$$

Assuming free entry into the R&D sector, the zero profit condition is

$$wH_A = P_A \underbrace{\gamma H_A^{\lambda} A^{1-\phi}}_{\dot{A}}. \tag{1.8.25}$$

Therefore, equation (8.25) can equivalently be written as,

$$wH_A = v\gamma H_A^{\lambda} A^{1-\phi}. \tag{1.8.26}$$

Because of the symmetry with respect to different intermediate goods, Barnett and Ghosh (2013) set $K = Ax$. The production function then is

$$Y = (AH_Y)^{1-\alpha}(K)^{\alpha}. \tag{1.8.27}$$

Hence, from equation (8.20) and (8.27), it follows that

$$w = (1-\alpha)A(\frac{K}{AH_Y})^{\alpha}. \tag{1.8.28}$$

From zero profits in the final goods sector, $\pi = H_Y{}^{1-\alpha}Ax^{\alpha} - pAx - wH_Y = 0$; and from equation (1.8.20), the following equation results

$$Y - wH_Y = pAx = \alpha Y. \tag{1.8.29}$$

Barnett and Ghosh (2013) note that wages equalize across sectors as a result of free entry and exit.

From the consumers' perspective, the agent's utility maximization problem is

$$\max_{c_t, \varepsilon_t} \frac{\int_t^{\infty} e^{-(\rho-n)t}\,[c(\tau)^{1-\sigma} - 1]}{1-\sigma} dt$$

subject to

$$\dot{K} = r_t[K_t + v_t A_t] + w_t H_t - c_t N_t - v_t \dot{A}_t - \dot{v}_t A_t,$$

$\dot{h}_t = \eta h_t{}^{\beta_1}(1-\varepsilon_t)^{\beta_2} - \theta g_A h_t$, and $\varepsilon_t \in [0,1]$,

where ρ is the subjective discount rate with $\rho > n > 0,$ and $\sigma \geq 0$ is the inverse of the intertemporal elasticity of substitution in consumption. Individuals choose consumption, c_t, and the fraction of time devoted to human capital production or to market work, ε_t.

In order to conduct bifurcation analysis, Barnett and Ghosh (2013) derive the following equations, which represent the dynamic equations for the model:

$$\frac{\dot{g}}{g} = \left(\frac{\alpha^2}{\sigma} - 1\right)m - \frac{\rho}{\sigma} + n + g + d\ , \tag{1.8.30}$$

$$\frac{\dot{m}}{m} = \frac{1-\alpha}{\alpha}[-\alpha^2 m + \alpha v + \phi(u-v)], \tag{1.8.31}$$

$$\frac{\dot{v}}{v} = (1-\alpha)m + v - g + \left\{\frac{(1-\alpha)\phi}{\alpha} - 1\right\}(u-v) - d, \qquad (1.8.32)$$

$$\frac{\dot{z}}{z} = \frac{1}{f(\beta_2-1)}\left[-z - \theta g_A(\beta_1 - 2) + \alpha v - \beta_2\frac{zvf}{u} - (1-\phi)(u-v) - n\right]$$

$$-(1-\beta_1)(z-\theta g_A), \qquad (1.8.33)$$

$$\frac{\dot{f}}{f} = \frac{1+f}{f(\beta_2-1)}\left[-z - \theta g_A(\beta_1 - 2) + \alpha v - \beta_2\frac{zvf}{u} - (1-\phi)(u-v) - n\right] \quad (1.8.34)$$

$$\frac{\dot{u}}{u} = z - \theta g_A + n - \phi(u-v) + \frac{1}{f(\beta_2-1)}\left[-z - \theta g_A(\beta_1 - 2) + \alpha v - \beta_2\frac{zvf}{u} -\right.$$

$$\left.(1-\phi)(u-v) - n\right]. \qquad (1.8.35)$$

According to Barnett and Ghosh's (2013) Definition 1, a steady state is a balanced growth path with zero growth rate. The steady state $s^* = (g^*, m^*, v^*, z^*, f^*, u^*)$ is derived by solving $\dot{g} = \dot{m} = \dot{v} = \dot{z} = \dot{f} = \dot{u} = 0$. The results are as follows:

$$z^* = \frac{n\theta}{\phi},$$

$$v^* = \frac{\rho - n}{\alpha} + \frac{n\sigma}{\phi\alpha},$$

$$u^* = v^* + \frac{n}{\phi},$$

$$m^* = \frac{v^*}{\alpha} + \frac{n}{\alpha^2},$$

$$g^* = (1 - \frac{\alpha^2}{\sigma})m^* + \frac{\rho}{\sigma} - n - d,$$

$$f^* = \frac{u^*}{v^*\beta_2}\left(\frac{\phi\rho}{\theta n} - \frac{(\phi + 1 - \sigma)}{\theta} - (\beta_1 - 1)\right).$$

Barnett and Ghosh (2013) derive the growth rate of technology to be $g_A = \frac{n}{\phi}$. The goal is to examine the existence of codimension 1 and codimension 2 bifurcations in the dynamical system defined by (1.8.30)-(1.8.35). The usual way to identify codimension-1 bifurcation is by varying a single parameter, while the usual way to identify codimension-2 bifurcation is by varying 2 parameters.

Barnett and Ghosh (2013) discuss reasons accounting for the occurrence of cyclical behaviors. The economic intuition behind the cycle phenomenon is described as follows. Suppose profits for monopolists increase. Then the price for designs, P_A, is bid up, since the market for designs is competitive. From (1.8.26), wages, w, in the R&D sector will rise. Higher wages lead to a shift of labor from output production to the research sector. Furthermore, the technological growth rate, g_A, will rise, if externalities to R&D are present. Assuming a negative effect of technical progress on human capital investment, i.e., $\theta > 0$, human capital accumulation, h_t, declines. According to (1.8.14) and (1.8.19), the price falls from a decline of average quality of labor. Monopoly profits then fall, completing the mechanism of this cycle.

Barnett and Ghosh (2013) use the numerical continuation package Matcont to detect Andronov-Hopf bifurcations. Table 1.8.2 reports the values of the subjective discount rate, ρ, the share of human capital, β_1, and the share of time devoted to the human capital production, β_2, the effect of technological progress on human capital accumulation, θ, and the depreciation rate of capital, d. Those parameters are treated as free parameters, at which Hopf bifurcation can occur.[24]

As discussed in section 1.8.2, a positive first Lyapunov coefficient indicates the existence of subcritical Hopf bifurcation. Therefore, since cases reported in Table 1.8.2 are associated with positive first Lyapunov coefficients, an unstable limit cycle with periodic orbit bifurcates from the equilibrium. When ρ, β_1, θ, and d are treated as free parameters, a slight perturbation of them gives rise to branch points (pitchfork/transcritical bifurcations).

Barnett and Ghosh (2013) investigate the stability properties of cycles generated by different combination of such parameters. The parameter, ρ, taken as a free parameter, gives rise to two period doubling (flip) bifurcations, one of which occurs at $\rho = 0.0257$ and the other at $\rho = 0.0258$. Both bifurcations have negative normal form coefficients, indicating stable double-period cycles.

Table 1.8.2 Stability Analysis of a Variant of Jones Semi-Endogenous Growth Model

Parameters varied	Equilibrium bifurcation	Continuation
β_1: $\{\alpha = 0.4, \rho = 0.055,\ \beta_2 = 0.04, n = 0.01, d = 0, \theta = 0.4, \phi = 1, \sigma = 8\}$	**Branch Point (BP)** $\beta_1 = 1$	
β_1: $\{\alpha = 0.4, \rho = 0.025772, \beta_2 = 0.04, n = 0.01, d = 0, \theta = 0.4, \phi = 0.8, \sigma = 0.08\}$	**Hopf (H)** First Lyapunov coefficient=0.0000230, $\beta_1 = 0.19$	
β_2: $\{\alpha = 0.4, \rho = 0.025772, \beta_1 = 0.19, n = 0.01, d = 0, \theta = 0.4, \phi = 0.8, \sigma =$	**Hopf (H)** First Lyapunov coefficient=0.00002302,	

[24] Table 1.8.2 is a replicate of Barnett and Ghosh (2013) Table 1.

0.08}	$\beta_2 = 0.040000$	
d: $\{\alpha = 0.4, \beta_1 = 0.19, \rho = 0.055, \beta_2 = 0.04, n = 0.01, \theta = 0.4, \phi = 1, \sigma = 8\}$	**Branch Point (BP)** $d = 0.826546$	
ρ: $\{\alpha = 0.4,\ \beta_1 = 0.19, \beta_2 = 0.04, n = 0.01, d = 0, \theta = 0.4, \phi = 1, \sigma = 0.08\}$	**Hopf (H)** First Lyapunov coefficient=0.0000149, $\rho = 0.025772$ **Branch Point (BP)** $\rho = 0.026726$ **Hopf (H)** Neutral saddle $\rho = 0.026698$	**Bifurcation of limit cycle** **Period doubling** (period=1569.64; $\rho = 0.0257$) Normal form coefficient=-4.056657e-013 **Period doubling** (period=1741.46; $\rho = 0.0258$) Normal form coefficient= -7.235942e-015 **Limit point cycle** (period=2119.53; $\rho = 0.0258$) Normal form coefficient=7.894415e-004 **Period doubling** (period=2132.13; $\rho = 0.0258$) Normal form coefficient=-1.763883e-013
θ: $\{\alpha = 0.4,\ \beta_1 = 0.19, \beta_2 = 0.04, n = 0.01, d = 0, \rho = 0.029710729,\ \phi = 0.69716983, \sigma = 0.08\}$	**Hopf (H)** First Lyapunov coefficient=0.0000230, $\theta = 0.40000$ **Hopf (H)** First Lyapunov coefficient=0.00001973, $\theta = 0.355216$	**Codimension-2 bifurcation** **Generalized Hopf(GH)** $\theta = 0.000044, \rho = 0.580853$, $\mathrm{L2} = 0.000001254$ **Bogdanov-Takens(BT)** $\theta = 0, \rho = 0.644247$ $(a, b) = (0.000001642, -0.003441)$ **Generalized Hopf(GH)** $\theta = 0.000055, \beta_1 = 0.584660$, $\mathrm{L2} = 0.0000008949$ **Bogdanov-Takens(BT)** $\theta = 0, \beta_1 = 0.903003$ $(a, b) = (0.000006407790,$

	Hopf (H) Neutral saddle, $\theta = 0.612624$ **Branch Point (BP)** $\theta = 0.613596$	0.03291344)

From further computation, Barnett and Ghosh (2013) find two limit cycles with different periods present near the LPC point at $\rho = 0.0258$ bifurcating from the Hopf point. They also find another period doubling (flip) bifurcation at $\rho = 0.0258$. Barnett and Ghosh (2013) then investigate the existence of codimension-2 bifurcations by first taking θ and ρ as free parameters and then taking θ and β_1 as free parameters. There are two types of codimension 2 bifurcations: Bogdanov-Takens and Generalized Hopf. At each Bogdanov-Takens point the system has an equilibrium with a double zero eigenvalue. The bifurcation point of the Generalized Hopf bifurcation separates branches of subcritical and supercritical Andronov-Hopf bifurcations in the parameter plane. The Generalized Hopf points are nondegenerate, since the second Lyapunov coefficient is nonzero. The system has two limit cycles for nearby parameter values, which collide and disappear through a saddle–node bifurcation.

1.9 Conclusion

At this stage of this research, we believe that Grandmont's conclusions appear to hold for all categories of dynamic macroeconomic models, from the oldest to the newest. So far, the findings we have surveyed suggest that Barnett and He's initial findings with the policy-relevant Bergstrom-Wymer model appear to be generic. We anticipate that further studies with other models will produce similar results, and advances in nonlinear and stochastic bifurcation are likely to find even deeper classes of bifurcation behavior, including perhaps chaos, which is precluded by linearization. This survey is designed to facilitate such future studies.

The practical implications of these findings include the following. (1) Policy simulations with macroeconometric models should be run at various points within the confidence regions about parameter estimates, not just at the point estimates. Robustness of dynamical inferences based on simulations only at parameters' point estimates is suspect. (2) Increased emphasis on measurement of variables is warranted, since small changes in variables can alter dynamical inferences by moving bifurcation boundaries and their distances from parameter point estimates. (3) While bifurcation phenomena are well known to growth model theorists, econometricians should take heed of the views of systems theorists, who have found that bifurcation stratification of the parameter space of dynamic systems is normal, and should not be viewed as a source of model failure or defect.

Chapter 2: Price Stickiness and Markup Variations in Market Games[25]

2.1 Introduction

Contemporary macroeconomic theory has been built on the three pillars of imperfect competition, nominal price rigidity, and strategic complementarity. The stickiness of prices (and wages in particular) is a well-established empirical fact, with early observations about the phenomenon dating back to Alfred Marshall. Because the friction of price stickiness cannot occur in perfectly competitive markets, modern micro-founded (e.g., New Keynesian) models have been forced to abandon the standard Arrow-Debreu paradigm of perfect competition in favor of models where agents may influence market prices. Strategic complementarity enters the picture as a mechanism for explaining the kinds of coordination failures that lead to sustained slumps like the Great Depression or the aftermath of the 2007-2008 financial crisis. Early work by Cooper and John (1988) lay out the importance of these three features for macroeconomics.

The need for imperfect competition becomes particularly transparent when one notes the importance of firms' markups of prices over marginal costs in allowing for quantity adjustments independently of price adjustments in response to market shocks. This is because prices equal marginal costs in competitive markets, and any variation in quantities must be accompanied by variations in prices. For models with nominal rigidities to work, some degree of positive markups is necessary. The role of markups in macroeconomic fluctuations has been examined closely by Rotemberg and Woodford (1991, 1992, 1999). These papers

[25] This paper is published as Chen et al. (2017).

have formed the basis for virtually all of the follow-on work in the new classical synthesis, and its reliance on imperfectly competitive market structures coupled with the dynamic structure of the neoclassical growth model. In most of this work, imperfect competition is introduced by imposing monopolistic competition via the Dixit and Stiglitz (1977) (hereinafter DS) model.

The DS model provides a simple and tractable way to model price-setting behavior in an otherwise competitive setting that strips away the sophistication of strategic behaviors that appear in settings of tight oligopoly. At the time of its introduction, the tractability of this model offset concerns over the empirical fact of oligopoly (in many industries including grocery retailing, banking, transportation, energy, telecommunications, and media), though in fairness to Dixit and Stiglitz, we note that their original model was one of preferences for diversity, rather than specialization in production.

Interestingly, there was another model around at that time which showed how to take explicit account of imperfect competition and large firms in a general equilibrium setting, the market game model developed by Shapley and Shubik (1977) and extended to production economies by Dubey and Shubik (1977). The Shapley-Shubik market game (hereinafter, market game) model received quite a bit of attention in the general equilibrium literature of the 1980’s and 1990’s, but was not considered as an alternative to models of monopolistic competition in macroeconomics. While the DS model certainly had important early adoption advantages (particularly in its first appearance as a model of production specialization in trade theory), the sophistication of the market game model likely deterred its applications, despite its distinct claim to being the best general equilibrium extension of well-known models in industrial organization, in the sense of following the original Nash framework for showing equilibrium in non-cooperative games (see, for example, Dubey and Geanakoplos 2003).

We believe the market game model can provide a significantly better micro-foundation for macroeconomics than do either the conventional real business cycle (hereinafter RBC) models based on the neoclassical growth model, or New Keynesian (hereinafter NK) models based on the DS model. Our belief is premised on the following three facts. First, as noted above, it is a simple empirical fact that modern industrial economies are populated by large firms that interact strategically across different markets in which they operate. These strategic interactions have been widely studied in partial equilibrium contexts in the theory of Industrial Organization (hereinafter IO), but macroeconomics has routinely ignored this branch of economics in favor of simpler models involving either perfect competition (RBC models) or models of local monopoly (the DS model). The market game model has similarly been overlooked, despite its potential of allowing for significant extension of findings in the IO literature to general equilibrium. From an empirical perspective, since the adoption of the monopolistic competition framework in macroeconomics, there has been a marked increase in industrial concentration. The President's Council of Economic Advisors Issue Brief (2016) documents concentration since the 1980's not only in technology industries (e.g. aerospace, microchip, operating system, software, and smart phone) but also in the traditional manufacturing and extractive industries, and in finance. As the report notes, some of this increase in concentration has been due to technological innovations and associated scale phenomena, and some has been due to mergers and acquisitions. Regardless of the cause, the new empiricism of market power suggests that economists should be paying more attention to the strategic interactions of large firms in oligopolistic market structures.

Second, oligopoly models allow the introduction of an additional strategic dimension beyond imperfectly competitive pricing markups, which makes possible equilibrating quantity adjustment processes – as we will show here – that do not require variation in prices, in some versions of the model. This is in marked contrast to the additional frictions required in DS-based models (menu costs or Calvo contracts) for price stickiness to occur. There are other papers that employ strategic models to incorporate or generate price stickiness. For example, Fershtman and Kamien (1987) study duopolistic competition in a model with a homogeneous good, and incorporate sticky prices by assuming that the desirability of the good is an exponentially weighted function of accumulated past consumption. Cellini and Lambertini (2007) extend Fershtman and Kamien (1987) by considering a dynamic oligopolistic game where goods are differentiated with sticky prices. Slade (1999) investigates the strategic implications of price adjustments, and empirically shows that strategic behavior aggravates price rigidity in a dynamic oligopoly. Both Carvalho (2006)'s model on heterogeneity in price stickiness and Fehr and Tyran (2008)'s model on limited rationality show that nominal rigidity prevails under strategic complementarity. Finally, Bhaskar (2002) provides a model of imperfect competition that produces a continuum of stable staggered price equilibria by introducing two levels of strategic interactions of firms within and across industries. Bhaskar (2002)'s model is the closest to ours in the sense that it shows how strategic interactions can end up generating price rigidities, albeit in terms of adjustment staggering rather than general nominal rigidity generated in our model.

Third, contemporary dynamic-stochastic-general-equilibrium (DSGE) models typically examine fluctuations in output, employment, and prices around a fixed steady state. This is done despite the fact that the data on business cycle fluctuations measure deviations in observed quantities from endogenously generated trend growth paths. Before macro models can be brought to data, then, the data itself must be detrended, usually based on ad hoc assumptions on the nature of economic growth. In RBC models, this is justified for the simple reason that long run growth in the neoclassical growth model must be assumed exogenously. In NK models, where (as Romer 1990 has shown) long-run growth is possible given the increasing returns to specialization inherent in the DS technology, macro applications of the model generally just ignore increasing returns and adopt the RBC practice of working with detrended data and stochastic steady states. The other well-known model of endogenous growth - the Schumpeterian model by Aghion and Howitt (1992) - has seen only minor applications at the intersection of IO and macroeconomics (see, for example, Aghion and Howitt 2000). The market game model, on the other hand, has the potential to allow for explicit consideration of growth in terms of its ability to accommodate increasing- returns-to-scale technologies, as well as the fact that it nests both DS and Aghion and Howitt (1992) models given the abstract specification of production activities in the model. While dealing explicitly with increasing-returns-to-scale technologies is more difficult than dealing with convex technologies, it is not intractable. In an earlier study, Korpeoglu and Spear (2016) extend the market game model with production to allow for increasing-returns-to-scale technologies, and show how imperfect competition in the market game remedies the standard problem that competitive firms operating under increasing-returns-to-scale technologies face of either wishing to produce infinite output, or, if restricted to marginal-cost pricing, needing to be subsidized to offset losses. This analysis also provides some weak results on the existence of equilibrium, though it should be no surprise that strong existence results are unattainable when technology dictates limits on the number of firms that can be active in equilibrium.

In this paper, we show that the market game generates equilibria that have two important features. First, we show that when firms have market power, their market-shares in both input and output markets affect the first-order conditions of their best responses, in ways that resemble the effects of price changes. From this observation, we are able to establish that firm quantity adjustments (holding input prices fixed) can maintain the Nash equilibrium of the model in versions of the model that exhibit indeterminacy of the Nash equilibrium. Hence, these versions of the model naturally admit sticky prices, regardless of the mechanism(s) that might lead firms to want to keep input prices unchanging. To the best of our knowledge, this is a new result. Second, we show that there is a close relationship between any individual firm's markup of price over marginal cost and its market share. As we noted above, the case for positive markups in macroeconomic models has been argued persuasively by Rotemberg and Woodford (1991, 1992, 1999). The relationship between markups and market shares, however, has not received attention, to the best of our knowledge. Rotemberg and Woodford (1992), for example, consider a model of oligopoly, but focus on symmetric Nash equilibrium in which each firm's market share is the same. This allows them to make predictions about how markups change as a response to demand or productivity shocks. What the market game brings to the discussion of markups that is new, is the fact that markets populated by finite numbers of firms operating under possibly different technologies will generate data on markup movements over different equilibria that can vary positively, negatively, variably, or not at all over business-cycle-like expansions and contractions. This is interesting in light of recent work by Nekarda and Ramey (2013) showing that "updated empirical methods and data" indicate that markups are weakly procyclic or acyclic, in contrast to the results found in the earlier work on markups and productivity co-movements.

The remainder of the paper is organized as follows. Section 2.2 lays out the basic market game model; Section 2.3 provides the detailed analysis of price stickiness and markup variations; and Section 2.4 concludes.

2.2 Model

We work initially with a standard market game model with production along the lines first considered by Dubey and Shubik (1977). In this section, we elaborate on the model ingredients. Most of our formulation of the model and our notation will follow that of Peck and Shell (1990) and Peck et al. (1992).

2.2.1 Agents

The economy consists of two types of agents: consumers ("she") and firms ("it"). There are $M < \infty$ consumers who are endowed with production inputs $\bar{e}_i \in \mathbb{R}_+^N$ and sell these inputs to firms that produce outputs from which consumers derive utility. For simplicity, we assume that consumers derive no direct utility from the consumption of their input endowments.

Preferences of consumers are defined over output goods vectors $x_i \in \mathbb{R}_+^J$. Utility functions are assumed to be at least twice continuously differentiable, strictly increasing, strictly concave, and satisfy Inada conditions. There are $K_j < \infty$ firms of finitely many types $j \in \{1,2,...,J\}$ that produce output good j using a production technology specified by a production function $q_{k_j}^j = f_{k_j}(\varphi_{k_j})$ where $\varphi_{k_j} \in \mathbb{R}_+^N$ is the vector of inputs for firm $k_j \in \{1,2,...,K_j\}$ in production sector $j \in \{1,2,...,J\}$, and each production function is twice continuously differentiable and strictly quasi-concave. We will denote the total number of firms by $\Im = \sum_{j=1}^{J} K_j$. We assume that consumers are exogenously endowed with ownership shares of each firm. Specifically, we let $\theta_i^{k_j}$ be consumer i's ownership share of firm k_j in sector j.

2.2.1.1 Firm actions

Firms purchase production inputs from consumers and use them to produce outputs, which they then sell back to consumers based on their expectations of prices they can receive for their outputs. Since firms are not endowed with production inputs, they must bid for these inputs on input trading posts (which are endemic to the market game). We assume that firms aim to maximize their profits.[26] We let p^j be the price of output good j, and r^n be the price of input good n, and $r = (r^1, r^2, ..., r^N)$ be the vector of input prices. The profit of firm k_j is then

$$\pi_{k_j} = p^j q_{k_j}^j - \sum_{n=1}^{N} r^n \varphi_{k_j}^n = p^j f_{k_j}(\varphi_{k_j}) - r\varphi_{k_j}. \tag{2.1}$$

Input prices are determined on input trading posts. We let $w_{k_j}^n$ denote firm k_j's bid on input trading post $n \in \{1,2,...,N\}$, and $w_{k_j} = (w_{k_j}^1, w_{k_j}^2, ..., w_{k_j}^N) \in \mathbb{R}_+^N$ denote firm k_j's vector of bids

[26] In the absence of perfect competition, it is well known that shareholders can disagree on the objective of the firm they own. As it is beyond the scope of this paper to justify that profit maximization is the correct objective for the firm operating under increasing-returns-to-scale technology, we just take this assumption as it is.

for inputs.[27] The aggregate bid at trading post n is $W^n = \sum_{j=1}^{J}\sum_{k_j=1}^{K_j} w_{k_j}^n$. As is standard, we let $W_{-k_j}^n$ denote the aggregate bid at trading post n except for the bid of firm k_j. Moreover, we let e_i^n denote consumer i's offer at input trading post n, and $E^n = \sum_{i=1}^{M} e_i^n$ denote the aggregate offer at input trading post n. Then, the price of input good n is then defined as $r^n = \frac{W^n}{E^n}$. Firm k_j's allocation of input good n is given by its own bid for the input divided by the price of the input

$$\varphi_{k_j}^n = \frac{w_{k_j}^n}{r^n} = w_{k_j}^n \frac{E^n}{W^n}. \tag{2.2}$$

This is just the standard market game rule that allocates each firm the same proportion of the aggregate offer of the input good as its bid is to the aggregate bid. Firms earn unit of account revenues from the sale of their outputs on trading posts for output goods. Given $q_{k_j}^j = f_{k_j}(\varphi_{k_j})$ for $j \in \{1,2,...,J\}$, firm k_j will offer all of its output on the output trading post j, so we can define the aggregate offer at trading post j as $Q^j = \sum_{k_j=1}^{K_j} q_{k_j}^j$. As before, we let $Q_{-k_j}^j$ denote the aggregate offer at trading post j except for the offer of firm k_j. Given the

[27] The fictional trading posts introduced by Shapley and Shubik (1977) are essentially a metaphor for flows of expenditure and product between traders. By collecting these flows for specific markets on "trading posts," it simplifies the actions of choosing demand and supply allocations and streamlines the exposition of the market game form. In equilibrium, though, it is only the flows of expenditure and product that matter, not where they take place.

price p^j for the output good j, firm k_j can spend $p^j q^j_{k_j}$ units of account on the purchase of input goods. Hence firm k_j faces the following budget constraint for its bids on inputs

$$\sum_{n=1}^{N} w^n_{k_j} = p^j q^j_{k_j}. \tag{2.3}$$

Note that substituting for $w^n_{k_j}$ from (2.2) into (2.3) yields

$$\sum_{n=1}^{N} \varphi^n_{k_j} r^n \le p^j q^j_{k_j} \Rightarrow \pi_{k_j} = p^j q^j_{k_j} - \sum_{n=1}^{N} \varphi^n_{k_j} r^n \ge 0,$$

which means that firm k_j's profit cannot be negative. If the firm's budget constraint (2.3) is not satisfied, then its input allocation is zero and all of its offers are confiscated.

2.2.1.2 Consumer actions

Consumers bid on trading posts for output goods. Because consumers derive no utility from the consumption of their input endowments, but receive income from selling these inputs, consumers will sell as much of their endowments as possible to firms. Consumer i's income from the sale of her input endowments, then, is given by $r \cdot e_i = \sum_{n=1}^{N} r^n e^n_i = \sum_{n=1}^{N} \frac{W^n}{E^n} e^n_i$, where the aggregate bid W^n on the input market n is determined by firms' production decisions. In addition to their income from the sale of input endowments, consumers also receive (exogenously given) shares of profits from firms they own, so that consumer i's total income is

$$y_i = \sum_{n=1}^{N} \frac{W^n}{E^n} e_i^n + \sum_{j=1}^{J} \sum_{k_j=1}^{K_j} \theta_i^{k_j} \pi_{k_j}. \tag{2.4}$$

Note that if we had a small number of consumers, given the arbitrary distribution of ownership shares across consumers, consumers might want firms they own to deviate from profit maximization in order to increase the value of their sales of input endowments. This failure of shareholder unanimity in models with imperfect competition is well known. As we do not provide any insight into this issue here, we will simply assume that consumers take the value of their endowment others and the value of their profit shares as given. This can be justified more rigorously by assuming that the number of consumers is much higher than the number of firms, so that the ratio $\frac{e_i^n}{E^n}$ in (2.4) is negligible, and that ownership of firms is diffusely distributed. We let b_i^j denote consumer i's bid on output trading post $j \in \{1,2,...,J\}$, and $b_i = (b_i^1, b_i^2, ..., b_i^J) \in \mathbb{R}_+^J$ denote consumer i's vector of bids for outputs. The aggregate bid at trading post j is $B^j = \sum_{i=1}^{M} b_i^j$. As above, we let B_{-i}^j denote the aggregate bid at trading post j except for the bid of consumer i. The price of output good j is then defined as the ratio of the total bid for the output good j to the total offer of the output good j, i.e., $p^j = \frac{B^j}{Q^j}$. Consumer i's allocation of output good j is given by her own bid for the output divided by the price of the output

$$x_i^j = \frac{b_i^j}{p^j} = b_i^j \frac{Q^j}{B^j}. \tag{2.5}$$

This is just the standard market game rule that gives each consumer the same proportion of the aggregate offer of the output good as her bid is to the aggregate bid. Consumer i faces the following budget constraint for bids on outputs

$$\sum_{j=1}^{J} b_i^j \leq y_i = \sum_{n=1}^{N} \frac{W^n}{E^n} e_i^n + \sum_{j=1}^{J} \sum_{k_j=1}^{K_j} \theta_i^{k_j} \pi_{k_j}. \qquad (2.6)$$

As with firms, if the consumer's budget constraint is violated, her allocation is zero and all of her others are confiscated.

2.2.2 Market Game and Nash Equilibrium

With these definitions and characterization of agents in the model, we can now formally define the market game Γ.

Definition 2.1 Consumer i 's strategy set is

$$S_i = \left\{ (b_i, e_i) \in \mathbb{R}_+^{2J} \mid e_i < \bar{e}_i \right\} \text{ for } i \in \left\{ 1, 2, ..., M \right\}.$$

Firm k_j 's strategy set is

$$S_{k_j} = \left\{ w_{k_j} \in \mathbb{R}_+^N \right\}.$$

The full strategy set that then defines the offer-constrained game $\Gamma(\hat{e})$ is

$$S = \times_{i=1}^{M} S_i \times_{k_j=1, j=1}^{K_j, J} S_{k_j}.$$

Definition 2.2 A Nash equilibrium of the simultaneous-move market game consists of consumers' bids for outputs, and firms' bids for inputs given expectations of other agents' actions such that

1. all agents' bids are best responses given their expectations of other agents' bids, of consumers' input offers, and of firms' output offers;

2. the best responses are consistent with all agents' expectations of other agents' actions.

While choosing its profit-maximizing bids, firm k_j takes the aggregate offer $E^n = \sum_{i=1}^{M} e_i^n$ as given, but takes other firms' (including those of other sectors) bids for inputs into account. Firm k_j maximizes its profit in (2.1) subject to the allocation rule in (2.2) and budget constraint in (2.3) given the input price $r^n = \frac{W^n}{E^n}$ and output price $p^j = \frac{B^j}{Q^j}$. Plugging constraints (2.2) and (2.3) into the objective (2.1), we obtain the following unconstrained profit maximization problem

$$\max_{w_{k_j}} \frac{B^j}{Q^j} f_{k_j}\left(\left[w_{k_j}^1 \frac{E^1}{W^1}, \dots, w_{k_j}^N \frac{E^N}{W^N} \right]\right) - \sum_{n=1}^{N} w_{k_j}^n \tag{2.7}$$

Note that (2.7) is firm k_j's best response to other firms' actions. Taking first-order conditions gives

$$\begin{aligned} &\frac{B^j}{Q^j} \frac{\partial f_{k_j}}{\partial \varphi_{k_j}^n} \left[\frac{E^n}{W^n} - \frac{w_{k_j}^n E^n}{\left(W^n\right)^2} \right] - \frac{B^j q_{k_j}^j}{\left(Q^j\right)^2} \frac{\partial f_{k_j}}{\partial \varphi_{k_j}^n} \left[\frac{E^n}{W^n} - \frac{w_{k_j}^n E^n}{\left(W^n\right)^2} \right] - 1 \\ &= \frac{B^j}{Q^j} \frac{\partial f_{k_j}}{\partial \varphi_{k_j}^n} \frac{E^n W_{-k_j}^n}{\left(W^n\right)^2} \frac{Q_{-k_j}^j}{Q^j} - 1 = 0 \end{aligned} \tag{2.8}$$

Plugging $p^j = \frac{B^j}{Q^j}$ and $r^n = \frac{W^n}{E^n}$ into (2.8) gives

$$\frac{p^j}{r^n}\frac{\partial f_{k_j}}{\partial \varphi^n_{k_j}}\frac{W^n_{-k_j}}{W^n}\frac{Q^j_{-k_j}}{Q^j} = 1 \tag{2.9}$$

Note that if the market contains a very large number of firms, ratios $\frac{W^n_{-k_j}}{W^n}$ and $\frac{Q^j_{-k_j}}{Q^j}$ in will be almost one, and hence (2.9) boils down to $p^j \frac{\partial f_{k_j}}{\partial \varphi^n_{k_j}} = r^n$, which states that the value of the marginal product of input good n is equal to the price of input good n.

While choosing her utility-maximizing bids, consumer i takes the aggregate output Q^j as given, but takes other consumers' bids for outputs into account. Consumer i maximizes her utility $u_i(x_i)$ subject to the allocation rule in (2.5) and budget constraint in (2.6). Plugging the constraint (2.5) into the objective yields the following budget-constrained utility maximization problem

$$\max_{b_i} u_i\left[b_i^1 \frac{Q^1}{B^1}, \ldots, b_i^J \frac{Q^J}{B^J}\right] \tag{2.10}$$

$$s.t. \sum_{j=1}^{J} b_i^j \le \sum_{n=1}^{N} \frac{W^n}{E^n} e_i^n + \sum_{j=1}^{J}\sum_{k_j=1}^{K_j} \theta_i^{k_j}\left[\frac{B^j}{Q^j} q^j_{k_j} - r \cdot \varphi_{k_j}\right]. \tag{2.11}$$

Note that (2.10)-(2.11) is consumer i's best response to other consumers' actions. Taking first-order conditions gives

$$\begin{aligned}&\frac{\partial u_i}{\partial x_i^j}\left[\frac{Q^j}{B^j}-\frac{b_i^j Q^j}{\left[B^j\right]^2}\right]+\lambda\left[\frac{\sum_{k_j=1}^{K_j}\theta_i^{k_j} f_{k_j}}{Q^j}-1\right]\\&=\frac{\partial u_i}{\partial x_i^j}\left[\frac{Q^j}{B^j}\frac{B_{-i}^j}{B^j}\right]+\lambda\left[\frac{\sum_{k_j=1}^{K_j}\theta_i^{k_j} f_{k_j}}{Q^j}-1\right]=0\end{aligned} \tag{2.12}$$

where λ is the Lagrange multiplier of consumer i's budget constraint in (2.11). Note that we do not need to consider the effect of a change in consumer i's bid on input prices because of the envelope theorem as applied to firms' profit maximization problems. Finally, note that if the market contains a very large number of firms, the total offer Q^j on output trading post j approaches infinity, and if the market contains a very large number of consumers, the ratio $\frac{B_{-i}^j}{B^j}$ approaches one. Then, the consumer first-order condition (2.12) boils down to what we get in the competitive limit, the ratio of marginal utility to the price is equal to the Lagrange multiplier, i.e. $\frac{\partial u_i / \partial x_i^j}{p^j}=\lambda$ where $p^j=\frac{B^j}{Q^j}$.

2.3 Analysis

In this section, we provide the analysis of coordination equilibria, price stickiness, and markup variations. Korpeoglu and Spear (2016) show conditions for the existence of a Nash equilibrium for a production market game with arbitrary returns-to-scale-technologies. Unlike in the case of strictly convex technologies, there are no strong existence results in the case of increasing-returns-to-scale technologies, for the simple reason that the non-negativity constraint on profits can become binding when there are many increasing-returns-to-scale firms in the market. For our purposes here, then, we will simply assume that there can be increasing-returns-to-scale firms in each production sector together with standard constant or decreasing-returns-to-scale firms, and that the Nash equilibrium associated with the aggregate input endowment E exists.

The result we present here is essentially just a comparative static result showing that if firms cannot (or do not wish to) vary input prices, they can accommodate shocks to production or demand via adjustments in output. This is conceptually no different from what occurs in competitive models. What is new with the market game is the fact that some of this accommodation can be achieved via adjustments in firm market shares on both input and output markets. In the presence of coordination indeterminacies, this new adjustment mechanism can give rise to novel equilibrium behavior in the model.

We will consider two variants of the model. The first is the standard, simultaneous-move Shapley- Shubik (1977) model with production (as laid out above). It is well-known that in pure exchange versions of this model, there are a continuum of Nash equilibria due to the fact that agents in the model choose both bids and offers. This choice is indeterminate, however, since the first-order conditions with respect to bids or offers are the same. In imperfectly competitive markets, one of the choices between bids and offers is redundant for individual agents, though it affects prices in the model via variations in market thickness. (See Peck and Shell 1991 or Peck et al. 1992 for details.) In the production model, no such indeterminacy is possible for the simple fact that consumers earn income from the sale of their endowments. Hence, in this version of the model, the comparative static result shows that any accommodation to shocks (with sticky prices) will necessarily involve some degree of involuntary unemployment of input resources. We conjecture that same form of neo-Keynesian coordination failure equilibria can be generated in the model, though not without a significantly more sophisticated (likely search-theoretic) micro-foundation for the input markets, which we do not pursue here.

In the second variant of the model, we introduce a real indeterminacy by allowing agents to short sell their endowments by offering more than they own, subject to the constraint that in equilibrium, they must buy back the short amount. This version of the model is based on one originally introduced by Peck and Shell (1990). Peck and Shell (1990) note that this version of the market game must be modified by changing the bankruptcy rules so that if any consumer doesn't satisfy her budget constraint, every consumer is forced to consume her endowment. This rule change is necessary because for very large short sales, the game "referee" may not be able to find an equilibrium using only the resources of non-bankrupt consumers. Since this variation on the production market game does not require any consumer to offer less than her full endowment, it is consistent with the non-cooperative incentive consumers in the production game have for earning income.

For both versions of the game, the starting point for our analysis is the individual firm's first- order conditions for profit maximization. For firm k_j, these can be written as

$$p^j\left(\frac{Q^j_{-k_j}}{Q^j}\right)D_\varphi f_{k_j} - \widehat{W}\widehat{W}^{-1}_{-k_j}r = 0\ .$$

The significance of writing the first-order conditions this way stems from the fact that variations in the terms $\widehat{W}\widehat{W}_{-k_j}^{-1}r$ and $p^j\left(\frac{Q^j_{-k_j}}{Q^j}\right)$ affect the firm's optimal choice, in the same way that variations in the input output prices do for perfectly competitive firms. Note also that $\frac{Q^j_{-k_j}}{Q^j} = 1 - \frac{q^j_{k_j}}{Q^j}$, where $\frac{q^j_{k_j}}{Q^j}$ is firm k_j's market share on the output market j, and $\frac{W^n_{-k_j}}{W^n} = 1 - \frac{w^n_{k_j}}{W^n}$, where $\frac{w^n_{k_j}}{W^n}$ is firm k_j's market share on the input market n. Since market shares can be varied independently of aggregate bids on output or expenditures on inputs, this suggests the possibility of altering firms equilibrium output quantities via adjustments in market shares without changing prices.

To analyze the possibility of price preserving perturbations in the market game, we note first that since we only have $(\Im - 1)N$ independent expenditure shares, we would need another N variables in order to make a full-rank perturbation of the system of equations consisting of firm first order conditions. We can pick up these variables by allowing for variations in the firms' aggregate expenditures on inputs (which we can think of as flexible inside money or credit in the market game setting, or as a monetary policy action in a macro interpretation of the model). To get a price rigidity result, we need to append the condition

$$r - \widehat{W}E = 0$$

(where E is the vector of aggregate input offers) to the firm first-order conditions, giving us a system of $\Im N + N$ equations in $\Im N$ variables. To get a full-rank perturbation of this system, then, we need an additional N variables. We cannot use the output market shares as these variables as they are not independent of firm output quantities. In a partial equilibrium setting, we might think of using the output prices themselves as variables (assuming there are at least as many output goods as there are inputs goods). But, in a general equilibrium setting, we need the output prices (or, equivalently, the aggregate bids of consumers for output goods) to ensure equilibrium in the output markets. This logic, then, gives us our first result.

Proposition 2.3.1 For the simultaneous-move market game, equilibrium responses to demand or technology shocks will generically (in production functions) require variations in prices.

The genericity argument here simply requires noting that if some shock left (say) input prices constant, then the firm would be moving up or down an expansion path homothetically. An arbitrarily small perturbation in the firm's production function will then destroy this homotheticity.

We can get a full rank perturbation that keeps input prices constant if we have an indeterminacy in consumers' aggregate offers, since this gives us the additional N variables we need. So, we now consider the Peck and Shell (1990) short-sale variant of the model that gives rise to indeterminacy. As we noted above, in the short-sale model, consumers are allowed to offer more than their total endowment on the input market, as long as they buy back the excess offer. This gives consumers the opportunity to affect market shares, which will matter if the input markets are strategic, i.e., if firms are not negligible.

As Peck and Shell (1990) note, the key to analyzing the short-sale version of the market game is the so-called offer-constrained game, in which consumers offers are fixed exogenously, and viewed as parameters the underlying game. The utility of the offer-constrained game stems from the fact that, with bid-offer indeterminacy, any equilibrium in the offer-constrained game will also be an equilibrium in the unconstrained game.

We define the offer-constrained game formally as follows. Consumer i's (offer-constrained) strategy set is

$$S_i(\hat{e}) = \left\{(b_i, e_i) \in \mathbb{R}_+^{J+N} \mid e_i = \hat{e}_i\right\} \text{ for } i = 1, 2, ..., M.$$

Firm k_j's strategy set is

$$S_{k_j} = \left\{w_{k_j} \in \mathbb{R}_+^N\right\}.$$

The full strategy set that then defines the offer-constrained game $\Gamma(\hat{e})$ is

$$S(\hat{e}) = \times_{i=1}^{M} S_i(\hat{e}) \times_{k_j=1, j=1}^{K_j, J} S_{k_j}.$$

A Nash equilibrium of the simultaneous-move offer-constrained market game consists of consumers' bids for outputs, and firms' bids for inputs given expectations of other agents' actions such that

1. all agents' bids are best responses given their expectations of other agents' bids, of consumers' input offers, and of firms' output offers;

2. the best responses are consistent with all agents' expectations of other agents' actions.

For the short-sale game, we need to modify the punishment (as in Peck and Shell 1990) to state that if any consumer violates her budget constraint, all agents allocations revert to their endowments. In the short-sale game, we would also formally modify the definition of the consumer's strategy set to require that $\hat{e}_i \geq \bar{e}_i$, where, as before, $\bar{e}_i$ is consumer i's endowment vector.

With these preliminaries, we can now show our main result.

Theorem 2.3.1 Generically, there exist solutions to the production side equilibrium equations in variables consisting of firm input wage bill shares, aggregate input quantities, and aggregate input bids, in a neighborhood of any given offer-constrained Nash equilibrium for the economy.

Proof: See Appendix 2.A.

Given that output prices variables are not used to equilibrate the production side of the economy, they will continue to serve their usual purpose in equilibrating the demand side of the model. On can apply conventional general equilibrium techniques to show a similar generic transversality result for this equilibration process, though since this is not germane to our results, we do not pursue it here.

2.3.1 Discussion

The most striking thing about the continuum of equilibria generated in the short-sale model is the fact that the economy is always at full employment of input resources. The theorem implies that variations in short-sale amounts will lead to adjustments in firm market shares (on both input and output markets), which can lead to non-trivial variations in total output. One could extend this result to a stochastic market game in which the short-sale offers varied according to some extrinsic random variable (i.e. a sunspot), as in Peck and Shell (1991). This would result in a stochastic general equilibrium for the model in which individual firm market shares are constantly changing. As we will show in the next section, this variation in market shares leads, in turn, to variations in the observed mark-ups that imperfectly competitive firms charge. In a heterogeneous returns-to-scale environment, then, one of the key relationships in New Keynesian macro models - the variation of mark-ups over the business cycle - will be disrupted.

2.3.2 Markup Variations

As we noted in Introduction, it is a stylized fact in NK macro models that markups vary counter- cyclically. In a recent study, as Nekarda and Ramey (2013) note, however, the estimation of marginal costs from available data is quite tricky, and early attempts to study markup variations over the business cycle ended up relying on theoretical relationships (based typically on DS-based NK models) for the specification of marginal costs. Nekarda and Ramey (2013) revisit the question of cyclical movements in markups using updated adjustments of inputs to production functions typically used in such studies, and using a combination of aggregate and manufacturing-specific data. Contrary to the conventional stylized fact, they establish that markups are unconditionally procyclic. Specifically, they found that monetary, government spending, and technology shocks lead to procyclical markups, and consumer demand shocks lead to slightly procyclical or acyclical markups. We will show in this section that the market game model also makes specific predictions about markup variation in response to (comparative static) expansions or contractions, but, because firms can exhibit heterogeneity in the returns-to-scale properties of their technologies, the aggregate observed markup variation can be quite different from that of any particular rm.

We start by writing the firm’s cost-minimization problem:

$$\min_{w} \iota \cdot w \quad \text{s.t.} \quad f(\widehat{W}^{-1}\widehat{E}w) \geq q\,,$$

where the vector $\iota = (1,1,...,1)$ is a sum vector. The first-order conditions of this problem are

$$\iota^T - \lambda Df^T \widehat{W}^{-2} \widehat{W}_{-k} \widehat{E} = 0\,,$$

which reduces to

$$r^T \widehat{W} \widehat{W}_{-k}^{-1} - \lambda Df^T = 0\,.$$

Since the Lagrange multiplier in the cost-minimization problem is just the marginal cost, if we assume that the production function is homogeneous of degree δ, then by direct calculation we have that

$$MC(q) = \lambda = \frac{1}{\delta q} r^T \widehat{W} \widehat{W}_{-k}^{-1} \varphi(q)\,.$$

To calculate the markup, we note from the first-order conditions for profit maximization that

$$\widehat{W} \widehat{W}_{-k}^{-1} r = p \frac{Q_{-k}}{Q} Df\,.$$

Combining profit maximization and cost minimization results, we have

$$\lambda = \frac{1}{\delta q} p \frac{Q_{-k}}{Q} Df \cdot \varphi(q) = p \frac{Q_{-k}}{Q}\,.$$

Hence, we obtain

$$\frac{p}{\lambda}=\frac{Q}{Q_{-k}}=\frac{1}{1-\frac{q_k}{Q}}.$$

This result shows that if firm k's market share increases, its markup also increases. Because the firm's equilibrium market share depends on its own and other firms' technologies, we have no a priori reason to believe that measures of average market shares (and hence observed average markups) move in any systematic way during expansions or contractions. In the next section, we show in two examples that individual firm market shares can increase or decrease as we move from low-input-use equilibrium to high-input-use equilibrium.

2.3.3 Example 1

In this section, we provide a simple example with two firms that use a single input to produce a single output good, and carry through the equilibration calculations. We let $L(=L_1+L_2)$ denote the exogenously given aggregate offer of the input (hereinafter, labor). Production functions of firm 1 and 2 are

$$q_1=f_1(L_1)=L_1^2 \text{ and } q_2=f_2(L_2)=\left[L_2-\overline{K}\right]^{\alpha}, \tag{2.13}$$

respectively, where $0<\alpha<1$ and $\overline{K}$ is a fixed real cost of production for firm 2. We let $Q(=q_1+q_2)$ denote the aggregate output. We also let w_i denote firm i's bid on labor and let $W(=w_1+w_2)$ denote the aggregate bid on labor. As in the model, the price of input is $r=W/L$, and the price output is $p=B/Q$. The input allocations of firm 1 and 2 are

$$L_1=\frac{w_1}{r}=\frac{w_1}{W}L \text{ and } L_2=\frac{w_2}{r}=\frac{w_2}{W}L, \tag{2.14}$$

respectively. Firms take the aggregate offer of labor L as given, but take the other firm's bid on labor into account. Firm i's best response to the other firm's action is a solution to the following pro t maximization problem:

$$\max_{w_i} \frac{B}{Q} q_i - w_i \text{ s.t. } w_i \leq \frac{B}{Q} q_i,\ L_i = w_i \frac{L}{W}, q_i = f_i(L_i)\,.$$

The first constraint requires nonnegative profits. The first-order condition of firm 1 is

$$\frac{B}{Q} f'_1(L_1)\left[\frac{L}{W} - w_1 \frac{L}{W^2}\right] + Bf_1(L_1)\left[-\frac{1}{Q^2}\right]\left[f'_1(L_1)\left(\frac{L}{W} - w_1 \frac{L}{W^2}\right)\right] - 1 = 0 \ . \quad (2.15)$$

Reorganizing (2.15), we get

$$\frac{B}{Q^2}\frac{w_2 L}{W^2} f_2(L_2) f'_1(L_1) - 1 = 0\,.$$

Substituting (2.13) and (2.14) yields

$$2\frac{BL^2\left[\left(\frac{w_2}{W}\right)L - \overline{K}\right]^{\alpha}}{Q^2}\frac{w_2}{W}\frac{w_1}{W^2} = 1\,.$$

We denote $s_i = w_i / W$ since w_i / W is firm i's share of the input. Plugging $r = W / L$ and $p = B / Q$ gives

$$2\frac{p}{r}\frac{L\left[s_2 L - \overline{K}\right]^{\alpha}}{Q} s_2 s_1 = 1\,. \quad (2.16)$$

By symmetry, the first-order condition of firm 2 is

$$\frac{B}{Q^2}\frac{w_1 L}{W^2}\left[f_1(L_1)f_2'(L_2)\right]-1=0 .$$

Substituting (2.13), (2.14), and $r=W/L$ and $p=B/Q$, we get

$$\alpha\frac{p}{r}\frac{s_1}{Q}\left[s_1 L\right]^2\left[s_2 L-\overline{K}\right]^{\alpha-1}-1=0 . \tag{2.17}$$

Solving (2.16) and (2.17) together, we obtain

$$2s_2\left[s_2 L-\overline{K}\right]=\alpha s_1^2 L .$$

If we let $\widehat{\alpha}=\frac{\alpha}{2}$, $s_2=s$ and $s_1=1-s$, this condition reduces to the following simple quadratic form $(1-\widehat{\alpha})Ls^2+\left[2\widehat{\alpha}L-\overline{K}\right]s-\widehat{\alpha}L=0$. Furthermore, if $\overline{K}=0$, it reduces to $s=\frac{\sqrt{\widehat{\alpha}}}{\sqrt{\widehat{\alpha}}+1}$. Substituting back into (2.16) or (2.17) will then determine what output price must be, given any equilibrium input levels including input price.

Thus, if $\overline{K}=0$ and $\alpha=0.5$, both firms make positive profits.

2.3.4. Example 2

In this section, we provide an example with three firms that use a single input (i.e., labor) to produce a single output. Interestingly, this example explicitly shows the existence of multiple coordination equilibria even in the one-input, one-output model (for similar examples, see Benhabib and Farmer 1994 and references therein). Production functions of firm 1, 2, and 3 are as follows:

Firm 1: $q_1 = f_1(L_1) = A \cdot L_1^2,$

Firm 2: $q_2 = f_2(L_2) = B \cdot L_2^{\alpha}, \quad 0 < \alpha < 1$

Firm 3: $q_3 = f_3(L_3) = C \cdot L_3.$

We next consider profit maximization problems of these firms. Firm i 's best response to other firms' actions is a solution to the following profit maximization problem:

$$\max_{w_i} \frac{B}{Q} q_i - w_i$$

$$\text{s.t. } \; w_i \leq \frac{B}{Q} q_i, \; L_i = w_i \frac{L}{W}, \; q_i = f_i(L_i).$$

The first constraint implies that profits cannot be negative. The first-order condition of firm 1 is

$$\frac{B}{Q} f_1'(L_1)\left(\frac{L}{W} - w_1 \frac{L}{W^2}\right) + Bf_1(L_1)\left(-\frac{1}{Q^2}\right) f_1'(L_1)\left(\frac{L}{W} - w_1 \frac{L}{W^2}\right) - 1 = 0$$

$$\Rightarrow \frac{p}{r} \frac{1}{QW}\left(f_2(L_2) + f_3(L_3)\right) f_1'(L_1)(w_2 + w_3) = 1 \tag{2.18}$$

Similarly, the first-order conditions of firm 2 and 3 are

$$\frac{p}{r} \frac{1}{QW}\left(f_1(L_1) + f_3(L_3)\right) f_2'(L_2)\left(w_3 + w_1\right) = 1 \tag{2.19}$$

$$\frac{p}{r}\frac{1}{QW}\big(f_1(L_1)+f_2(L_2)\big)f_3{'}(L_3)\big(w_2+w_1\big)=1 \tag{2.20}$$

respectively. We denote the share of input by $s_i = w_i / W$, where $i \in \{1,2,3\}$. The aggregate output is given by $Q = A\cdot(s_1L)^2 + B\cdot(s_2L)^\alpha + C\cdot s_3L$. Substituting s_i and Q into (2.18), (2.19), and(2.20),we obtain the following conditions for firms 1, 2, and 3, respectively:

$$\frac{p}{r}\frac{\big(B(s_2L)^\alpha + Cs_3L\big)\cdot 2ALs_1(s_2+s_3)}{A(s_1L)^2 + B(s_2L)^\alpha + Cs_3L} = 1$$

$$\frac{p}{r}\frac{\big(A(s_1L)^2 + Cs_3L\big)\cdot B\alpha(s_2L)^{\alpha-1}(s_1+s_3)}{A(s_1L)^2 + B(s_2L)^\alpha + Cs_3L} = 1$$

$$\frac{p}{r}\frac{\big(A(s_1L)^2 + B(s_2L)^\alpha\big)\cdot C(s_1+s_3)}{A(s_1L)^2 + B(s_2L)^\alpha + Cs_3L} = 1.$$

We can solve for the equilibrium shares from the following three equations in three variables:

$$(2ABs_1(s_2+s_3)s_2^\alpha - \alpha ABs_1^2(s_1+s_3)s_2^{\alpha-1})\cdot L^{\alpha+1} + 2ACs_1s_3(s_2+s_3)\cdot L^2 - \alpha BCs_3s_2^{\alpha-1}(s_1+s_3)L^\alpha = 0$$

$$2ABs_1(s_2+s_3)s_2^\alpha L^{\alpha+1} + (2ACs_1s_3(s_2+s_3) - AC(s_2+s_3)s_1^2)\cdot L^2 - BCs_2^\alpha(s_1+s_2)L^\alpha = 0$$

$$s_1+s_2+s_3=1$$

Figure 2.1 demonstrates equilibrium shares for three equilibria that occur under $A = B = C = 1$, and $\alpha = 0.5$ as the aggregate input L varies. Interestingly, in each equilibria, the share of at least one firm approaches zero as the aggregate input L gets large. This, in turn, leaves three possible equilibrium industrial organization modes: i) monopoly with the decreasing-returns-to-scale firm in equilibrium 2 of Figure 2.1, ii) stable duopoly with increasing- and constant-returns-to-scale firms in equilibrium 1 of Figure 2.1, and iii) stable duopoly with decreasing- and constant-returns-to-scale firms in equilibrium 3 of Figure 2.1. In equilibrium 2 of Figure 2.1, increasing-returns-to-scale and constant-returns- to-scale firms both make positive profits regardless of the market thickness. However, when the market is thin (i.e., L is small), the decreasing-returns-to-scale firm makes positive profit; and when the market is thick (i.e., L is large), the decreasing-returns-to-scale firm makes negative profit. However, in equilibrium 2 of Figure 2.1, when the market is thick, the decreasing-returns-to- scale firm dominates the market while incurring negative profit. Thus, for all three firms to make positive profits in equilibrium 2, the market must be sufficiently thin (i.e., L is sufficiently small). In equilibrium 1 of Figure 2.1, all three firms make positive profits regardless of the thickness of the market. In equilibrium 3 of Figure 2.1, the increasing-returns-to-scale firm always makes negative profits with very small and diminishing market share, so it is likely that it will eventually exit the market, and the other two firms will share the market and earn positive profits. In this example, we can still obtain the multiplicity result if all three firms make positive profits (equilibrium 1 and 3 when market is sufficiently thin).

Figure 2.2 illustrates equilibrium shares for two equilibria that occur under $A=1, B=2, C=3, \alpha=2/3$, and in both equilibria the decreasing-returns-to-scale and constant-returns-to scale firms make positive profits. In equilibrium 1 of Figure 2.2, when the market is thin (i.e., L is small) the increasing- returns-to-scale firm has negative profits, and when the market is thick (i.e., L is large) the increasing-returns-to-scale firm makes positive profit. The equilibrium 2 of Figure 2.2 is similar to the equilibrium 3 of Figure 2.1 in the sense that the increasing-returns-to-scale firm always makes negative profit, and has diminishing market shares and it is likely to exit the market eventually, while the other two firms share the market and earn positive profits.

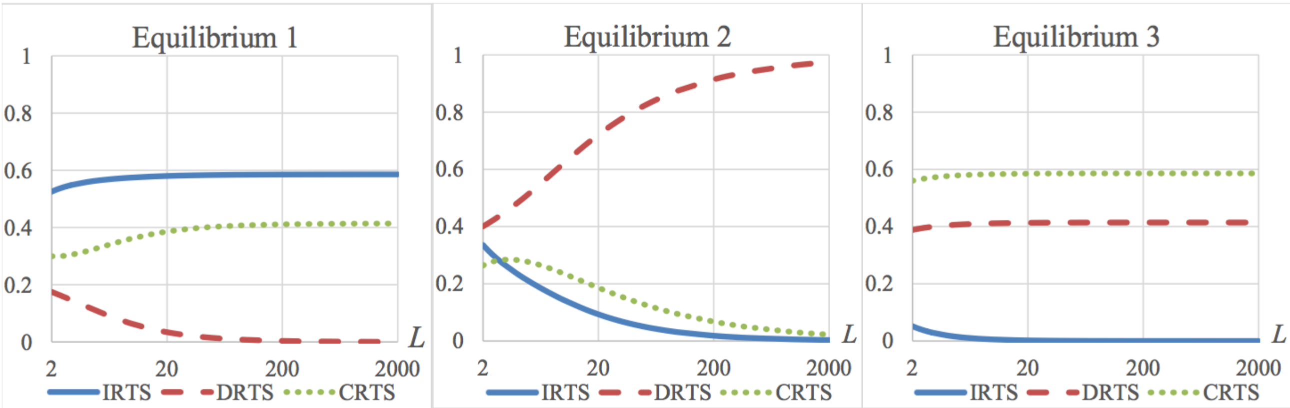

Figure 2.1 Input share allocations across increasing-returns-to-scale(IRTS), decreasing-returns-to-scale(DRTS), and constant-returns- to-scale(CRTS) firms as the aggregate input L varies in logarithmic scale. Setting: $A = B = C = 1$, $\alpha = 0.5$.

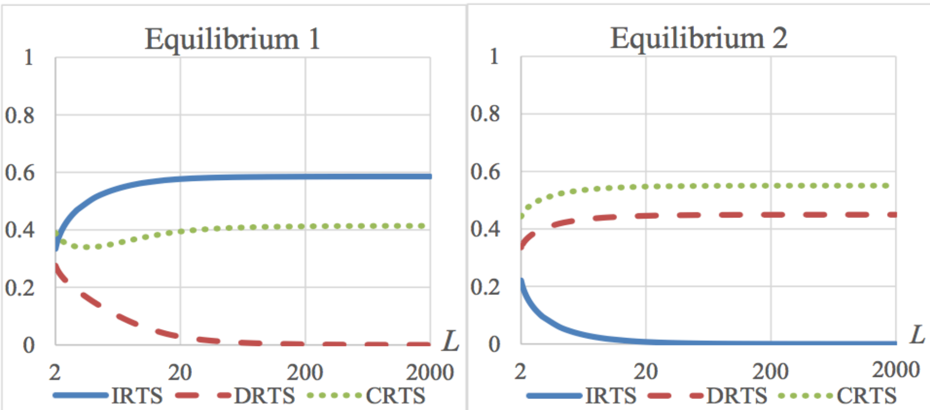

Figure 2.2 Input share allocations across increasing-returns-to-scale (IRTS), decreasing-returns-to-scale(DRTS), and constant-returns- to-scale(CRTS) firms as the aggregate input L varies in logarithmic scale. Setting: $A = 1, B = 2, C = 3, \alpha = 2/3$.

The two examples above show that thin market equilibria may generate negative profits while thick market does not, and thick market equilibria may also generate negative profits while thin market does not (in which case the dominant firm will eventually exit the market). We also observe that if the increasing-returns-to-scale firm has very small market share, it makes negative profit and is likely to exit the market, while the other two firms make non-negative profits, and are likely to share the market. If the decreasing-returns-to-scale firm dominates the market, it makes negative profit; if it does not dominate the market, which is when the market is sufficiently thin, all three firms make nonnegative profits. The possibility of profits being negative in these examples reflect the fact that these examples do not calculate the full Nash equilibria for the model, but rather only the firms' responses to variations in the input to production, holding the input prices constant. In a full Nash-equilibrium calculation, firms facing negative profit would need to make an exit decision, with the final equilibria then being based on a smaller number of active firms in the market.

2.4 Discussion and Conclusion

We have shown that variants of the Shapley-Shubik market game model with production can generate an equilibration mechanism that can lead to multiple coordination equilibria when the number of active firms is small. The equilibration process can accommodate nominal price rigidities, without any need for enforcing menu costs or other additional restraints on price adjustment. We also explicitly show the relationship between a typical firm's markup of price over marginal cost and its market share. The model itself is silent on what might cause price rigidities, and how different mechanisms (e.g., menu costs, search frictions, and learning) might interact with the basic model. We believe there are some interesting arguments in favor of learning and evolutionary dynamics that arise from the general equilibrium considerations in our analysis.

The problems with finding effective mechanisms for implementing equilibrium prices in competitive economies are well known. Scarf (1960)'s example shows that the presence of strong income effects can make simple price adjustment dynamics like the Walrasian tatonnement process ineffective. While the market game does provide an explicit price formation mechanism via the ratio of expenditure flows to quantity flows, Kumar and Shubik (2004) show that the market game is not immune to Scarf (1960)-like problems for simple adjustment dynamics akin to tatonnement.

On the other hand, there are a series of strong results in the literature on evolutionary game theory showing that when the Nash equilibrium to a game is strict (i.e., when the equilibrium is in pure strategies), then fitness-based (replicator) dynamics in which better responses to other agents' play are imitated lead to convergence to the Nash equilibrium. These results have not received much attention in the conventional general equilibrium analysis or related work in macroeconomics because of the time-scales on which these dynamics operate, and the often non-market-based nature of the interactions generating the convergence.

What the evolutionary game theory results do suggest (particularly in light of the fundamental problems introduced by income effects) is that equilibrium (either Nash or competitive) is something that must be learned rather than mechanically implemented. To the extent that Nash equilibria of the market game are evolutionarily stable, i.e., immune to deviations from Nash equilibrium strategies, the learning costs will be quite high since pricing experiments themselves become costly. Hence, the relatively more complex nature of evolutionary learning, as opposed to simple mechanical price adjustment processes, makes attaining an equilibrium costly, and provides an incentive for maintaining equilibrium prices once they are learned.

From a less heterodox perspective, the literature on search and matching, based on the seminal work of Burdett and Judd (1983), is capable of generating both price stickiness and staggered price adjustment in otherwise conventional economic models. This framework, particularly at the interface between wholesale and retail intermediaries could easily be adapted to the model we have presented here.

One thing that is clear from this discussion is that further work embedding the market game with production in a dynamic quantitative setting is worth undertaking.

Chapter 3: Endogenous Business Cycles in the Overlapping Generations Market Game Model[28]

3.1 Introduction

In this paper, we study whether strategic interactions contribute to instabilities of the economic dynamics in the overlapping generations (OLG) version of the Shapley-Shubik market game model with production (see, for example, Dubey and Shubik (1977), or Chen et al (2017)).

The study of complex dynamics in economic models has focused historically on the question of whether modern capitalist economies are inherently stable or unstable. The argument for instability is based on the fact of business cycles, which first appear historically with the onset of industrialization in the early 1800's in the West. The argument for stability first appears in early real business cycle models where exogenous aggregate shocks generate deviations from an otherwise stable steady-state of the model.

Grandmont (1985) was one of the first papers to raise the possibility that endogenous complex dynamics might provide an alternative explanation for business cycle fluctuations, in other words, the possibility that business cycles could be generated endogenously, that is, without making the assumption that the economy would never fluctuate in the absence of exogenous aggregate shocks. Grandmont shows that such dynamics could arise in conventional OLG models, although only for the case of sufficiently large risk aversion on the part of old agents in the model. Shortly after Grandmont's work appeared, Boldrin and

[28] This paper is Chen (2018).

Montrucchio (1986) showed that complex dynamics could also arise in the neoclassical capital model. In both of these modeling approaches, however, the parameter values required for chaotic trajectories to arise were unrealistic. For OLG models, Grandmont required relative risk aversion coefficients in excess of 8, while in the capital model, discount factors were required to be less than 0.35. Neither of these assumptions seemed at all realistic. Goenka et al. (1998) showed in the context of a pure exchange OLG market game that the nonlinearities introduced by imperfect competition were such that one could obtain chaotic dynamics even for log utility, as long as markets were thin in terms of the amount of endowment agents offered. Goenka et al. note that extensions of their work with this kind of model suggests that production smooths the model in the sense that complex dynamics are not as easily generated as in the pure exchange model. In this paper, we show that Goenka et al.'s observation is true.

Specifically, we will show in the paper that when incorporating production in the market game OLG model, the price dynamics depend on market thickness, general equilibrium price ratios, individual offers and particular choices of utility functions. Unlike standard CRRA utility functions assumed in Grandmont (1985) and Goenka et al. (1998), for complex dynamics to occur, the preferences in our model must be a mix of preferences, for example, a combination of preferences with constant relative risk aversions and increasing relative risk aversions. Endowment assumptions and market thinness alone cannot ensure the existence of complicated dynamics. In our paper, we also show impossibility of such price dynamics to occur for log-linear preferences. In other words, the case for complex dynamics to occur with particular production functions and utility functions is much more limited. As a result, complex dynamics are not as easily observable as in models without production.

Finally, we are able to confirm the results from Goenka et al. (1998) on the Pareto rankability of Nash equilibria in terms of market thickness under a stricter assumption, which has important welfare implications for business cycle-like activity based on the coordination equilibria that can arise in market game models (again, see Chen et al. (2017) for details on these equilibria).

The remainder of the paper is organized as follows. Section 2 specifies the model and market equilibrium. Section 3 displays the dynamics analysis. Section 4 studies the special case when the preferences are log-linear. Section 5 concludes. Some proofs are contained in the appendices.

3.2 The model

3.2.1 Agents

We consider a market game model of m firms with a single type of input goods and a single type of output goods. In an OLG setting, in each period there are n young agents and n old agents. The old own the firms that produce the same type of consumption goods. The young agents are endowed with labor and offer an exogenously fixed amount of labor. In each period, the young agents offer labor, make bids on consumption goods and purchase share ownerships of the two firms. The old would purchase consumption goods funded by selling their shares of m firms and profits of m firms.

Assume consumer i born time t, would offer labor $l_{i,t}$ when young. At time $s = t, t+1$, consumer i's bid for consumption goods is denoted $b_{i,s}^{t}$. The share of firm k purchased by consumer i is $a_{i,t}^{k}$ and

$$\sum_{i=1}^{n} a_{i,t}^{k} = 1. \tag{3.1}$$

Let the asset price for firm k be $q_{k,t}$, $k = 1,2,...,m$. The aggregate bid on output good is the sum of bids from young agents and bids from old agents, which is

$$B_t = \sum_{i=1}^{n} (b_{i,t}^{t} + b_{i,t}^{t-1}). \tag{3.2}$$

Let the aggregate output at time t be Q_t. The price of output good at time t is

$$p_t = \frac{B_t}{Q_t}. \tag{3.3}$$

The consumption of consumer i at time $s = t, t+1$ is

$$x_{i,s}^{t} = \frac{b_{i,s}^{t}}{p_s} = \frac{b_{i,s}^{t} \cdot Q_s}{B_s} \quad . \tag{3.4}$$

The aggregate input good is

$$L_t = \sum_{i=1}^{n} l_{i,t} \ .$$

Let firm k 's bid for input good be $w_{k,t}$. The aggregate bid on input good is

$$W_t = \sum_{j=1}^{m} w_{j,t}. \tag{3.5}$$

The input good price is thus given by

$$r_t = \frac{W_t}{L_t}. \tag{3.6}$$

Let the production function for firm k be $f_k(x)$ with input $L_{k,t}$, $k = 1,2$. The profit function for firm k at time t is given by

$$\pi_{k,t} = p_t f_k(L_{k,t}) - r_t L_{k,t}, \qquad k = 1,2.$$

3.2.2 Strategic interactions

3.2.2.1 Consumers side

At time t, consumer i uses labor income to purchase consumption goods and share ownerships of two firms. At time $t+1$, consumer i sells his share ownerships of two firms, and his consumption at time $t+1$ is funded by the time $t+1$ profits of the shares of the firms he owns and selling his shares of two firms.

We also make the following assumptions on consumers' utility function

Assumption 3.2.1 We assume

1. Utility is additively separable: $u(x_t^t, x_t^{t+1}) = U(x_t^t) + V(x_{t+1}^t)$.

2. U and V are smooth strictly increasing, strictly concave and satisfy Inada conditions.

Consumer i's utility maximization problem is thus given as follows:

$$\max_{\{x_{i,t}^t, x_{i,t+1}^t, a_{i,t}^1, \ldots, a_{i,t}^m\}} U(x_{i,t}^t) + V(x_{i,t+1}^t)$$

subject to

$$p_t x_{i,t}^t + \sum_{j=1}^{m} a_{i,t}^j q_{j,t} = r_t l_{i,t} \tag{3.7}$$

and

$$p_{t+1} x_{i,t+1}^t = \sum_{j=1}^{m} (q_{j,t+1} + \pi_{j,t+1}) a_{i,t}^j \tag{3.8}$$

In our model, consumers are assumed to be perfectly competitive, that is they take consumption good price p_t and asset prices $q_{k,t}$, $k = 1,2,...,m$ as given.[29] The input good price r_t is not affected by their optimization behavior, and hence it is taken as given by consumers. Firms' profits $\pi_{k,t+1}$, $k = 1,2$ are taken as given too.

An alternative form of utility maximization problem is as follows:

$$\max_{\{b_{i,t}^t, b_{i,t+1}^t, a_{i,t}^1, \ldots, a_{i,t}^m\}} U\left(\frac{b_{i,t}^t}{B_t} \cdot Q_t\right) + V\left(\frac{b_{i,t+1}^t}{B_{t+1}} \cdot Q_{t+1}\right)$$

subject to $$b_{i,t}^t + \sum_{j=1}^{m} a_{i,t}^j q_{j,t} = r_t l_{i,t}$$

and

[29] We assume asset prices are not all zeros, otherwise it would be akin to allowing the old to make bequests to their children, which then turns the model in one with an infinite-lived dynasty, rather than the OLG structure.

$$b_{i,t+1}^{t} = \sum_{j=1}^{m} (q_{j,t+1} + \pi_{j,t+1}) a_{i,t}^{j} .$$

Two forms of utility maximization problem are essentially identical except for a change of variables.

3.2.2.2 Firms side

At time t, firms will take the aggregate offer of labor L_t as given but will take account of the other firm's bid for labor. Firm i's best response to the other firm's action is determined by the solution to the optimization problem as follows:

$$\max_{w_{i,t}} \frac{B_t}{Q_t} f_i(L_{i,t}) - w_{i,t}$$

subject to

$$w_{i,t} \le \frac{B_t}{Q_t} f_i(L_{i,t})$$

$$L_{i,t} = w_{i,t} \frac{L_t}{W_t}, \; i = 1,2,...,m .$$

The constraint just implies that the net profit should be nonnegative. The profits of two firms at time $t+1$ are given respectively as follows

$$\pi_{1,t+1} = p_{t+1} f_1 \left(\frac{w_{1,t+1}}{W_{t+1}} L_{t+1}\right) - w_{1,t+1}$$

$$\pi_{2,t+1} = p_{t+1} f_2 \left(\frac{w_{2,t+1}}{W_{t+1}} L_{t+1}\right) - w_{2,t+1}$$

3.2.3 Best responses

3.2.3.1 Consumers

Replacing $x_{i,t}^t$ and $x_{i,t+1}^t$ in terms of $a_{i,t}^k$, $k=1,2,$ the two-period utility function can be written as

$$U\left(\frac{r_t l_{i,t} - \sum_{j=1}^{m} a_{i,t}^j q_{j,t}}{p_t}\right) + V\left(\frac{\sum_{j=1}^{m}(q_{j,t+1}+\pi_{j,t+1})a_{i,t}^j}{p_{t+1}}\right).$$

The best response functions for consumer i are solved from the following F.O.Cs:

F.O.Cs for $a_{i,t}^j$, $j=1,2,...,m$:

$$\frac{U'(x_{i,t}^t)}{V'(x_{i,t+1}^t)} = \frac{q_{j,t+1}+\pi_{j,t+1}}{q_{j,t}} \cdot \frac{p_t}{p_{t+1}}. \qquad (3.9)$$

According to (3.9), for $\forall j,l = 1,2,...,m,$ we also have

$$\frac{q_{j,t+1}+\pi_{j,t+1}}{q_{j,t}} = \frac{q_{l,t+1}+\pi_{l,t+1}}{q_{l,t}}$$

3.2.3.2 Firms

Firm i 's best response to the other firm's action is determined by the solution to the optimization problem at time t as follows

$$\max_{w_{i,t}} \frac{B_t}{Q_t} f_i(w_{i,t}\frac{L_t}{W_t}) - w_{i,t}$$

subject to

$$w_{i,t} \le \frac{B_t}{Q_t} f_i(w_{i,t}\frac{L_t}{W_t})$$

The constraint just implies that the net profit should be nonnegative. The F.O.C of the optimization problem of firm i with respect to $w_{i,t}$ shows:

$$\frac{B_t}{Q_t} f_i'(L_{i,t})\left(\frac{L_t}{W_t} - w_{i,t}\frac{L_t}{W_t^2}\right) + B_t f_i(L_{i,t})\left(-\frac{1}{Q_t^2}\right)\cdot\left(f_i'(L_{i,t})\left(\frac{L_t}{W_t} - w_{i,t}\frac{L_t}{W_t^2}\right)\right) - 1 = 0\,,$$

or
$$\frac{p_t}{r_t}\cdot f_i'(L_{i,t})\cdot\frac{Q_{-i,t}}{Q_t}\frac{W_{-i,t}}{W_t} = 1\,, \tag{3.10}$$

where $Q_{-i,t} = Q_t - f_i(L_{i,t})$ and $W_{-i,t} = W_t - w_{i,t}$.

Since there are m firms, there are m F.O.Cs like (3.10). Since production functions are assumed as known, the only variables in those m equations are input shares $\frac{w_{i,t}}{W_t}$, $i = 1,2,...,m$. Derived from (3.10), we also have the following $m-1$ equations, for $j = 2,3,...,m$:

$$f_1'(L_{1,t})\cdot\frac{Q_{-1,t}}{Q_t}\frac{W_{-1,t}}{W_t} = f_j'(L_{j,t})\cdot\frac{Q_{-j,t}}{Q_t}\frac{W_{-j,t}}{W_t}\,. \tag{3.11}$$

Since $\sum_{i=1}^{m} \frac{w_{i,t}}{W_t} = 1$, there are only $m-1$ variables in the above $m-1$ equations, thus we are likely to solve for input shares $\frac{w_{i,t}}{W_t}$, $i = 1,2,...,m$, from these $m-1$ equations. Substituting the input shares back to (3.10), we see that (3.10) determines the price ratio $\frac{p_t}{r_t}$, given the production function forms.

3.2.4 Market clearing condition

3.2.4.1 Goods market

The market clear condition is

$$\sum_{i=1}^{n} (x_{i,t}^{t} + x_{i,t}^{t-1}) = \sum_{j=1}^{m} f_j(L_{j,t}) = Q_t \ . \tag{3.12}$$

3.2.4.2 Money market

Assume the money supply in the market is $\overline{M}$, then

$$B_t + \sum_{j=1}^{m} q_{j,t} = \overline{M} \ . \tag{3.13}$$

The reason we only consider the consumer side is because money circulates between firms and agents, while the old agents own the firms. Firms pay wages to the young agents since they provide labor. The profit that amounts to $B_t - W_t$, along with the amount of money equal to $\sum_{j=1}^{m} q_{j,t}$ from sales of assets, is shared by old agents since they own the firms. So the old agents earn in total $B_t - W_t + \sum_{j=1}^{m} q_{j,t}$. Young agents give back all their wages W_t to the firms for consumption and share ownerships. Old agents also give back their earnings to the firms to purchase consumption. So in total the amount of money in circulation is equal to $W_t + \left(B_t - W_t + \sum_{j=1}^{m} q_{j,t} \right) = B_t + \sum_{j=1}^{m} q_{j,t}$. Hence we have (3.13).

3.2.4.3 Asset market

The asset market clearing condition is equation (3.1):

$$\sum_{i=1}^{n} a_{i,t}^{k} = 1, \ k = 1,2,...,m.$$

3.2.5 Market equilibrium

The market equilibrium is defined as follows

Definition 2.2.1 The Nash equilibrium for the market game OLG model is a sequence of bids $\left\{ b_{i,t}^{t}, b_{i,t}^{t-1} \right\}_{t=1,2,...}^{i=1,2,...,n}$, $\left\{ w_{j,t} \right\}_{t=1,2,...}^{j=1,2,...,m}$, $\left\{ q_{k,t} \right\}_{t=1,2,...}^{k=1,2,....,m}$ and $\left\{ a_{i,t}^{k} \right\}_{i=1,2,....,n}^{k=1,2,...,m}$ [30] such that

1. labor inputs are exogenously given by the sequence $\left\{ l_{i,t} \right\}_{t=1,2,....}^{i=1,2,...,n}$;

2. every agent and every firm's bids and shares of firms are a best response to the bids of other agents and other firms when those bids and shares of firms are taken as given;

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[30] $\left\{ a_{i,t}^{k} \right\}_{i=1,2,....,n}^{k=1,2,...,m}$ satisfies asset market clearing condition (1).

3. both money market and good markets are clear ;

4. the aggregate money supply $\overline{M}$ is also exogenously given.

3.3 Equilibrium dynamics

For this part of analysis, we make the following assumption

Assumption 3.3.1 All the exogenous labor offers are identical and independent of time, i.e. $l_{i,t} = l_i = l$ for all i and t.

Assumption 3.3.2 Agents born at time t are identical.

These assumptions together with the stationarity of population imply that the aggregate labor input is given by $nl = L$. Since all agents born at the same period are identical, it makes sense to consider a symmetric equilibrium in which asset shares are also identical, that is, $a_{i,t}^k = \frac{1}{n}$, $i = 1,2,...,n$, $k = 1,2,...,m$. Given the specific production function forms, we also have

1. according to (3.11), the input labor shares are independent of time, so we denote $\frac{w_{j,t}}{W_t} = s_j$;

2. according to (3.10), $\frac{p_t}{r_t}$ is independent of time, so we denote $\frac{p_t}{r_t} = \frac{p}{r}$; also we notice $\frac{r}{p}$ is a function of L, we denote this function $g(L)$.

3. for each firm, the output $f_j(s_j L)$ is independent of time, hence the aggregate output $Q=\sum_{j=1}^{m} f_j(s_j L)$ is also independent of time.

Therefore equation (3.7) and (3.8) are reduced to

$$x_{i,t}^{t}=\frac{r}{p}\cdot l-\frac{1}{n}\sum_{j=1}^{m}\frac{q_{j,t}}{p_t} \tag{3.14}$$

and

$$\begin{aligned} x_{i,t+1}^{t} &= \frac{1}{n}\sum_{j=1}^{m}\frac{q_{j,t+1}+\pi_{j,t+1}}{p_{t+1}} \\ &= \frac{1}{n}\sum_{j=1}^{m}\left(\frac{q_{j,t+1}}{p_{t+1}}+f_j(s_j L)-\frac{r_{t+1}}{p_{t+1}}\cdot s_j L\right) \\ &= \frac{1}{n}\sum_{j=1}^{m}\frac{q_{j,t+1}}{p_{t+1}}+\frac{Q}{n}-\frac{r}{p}\cdot l \end{aligned} \tag{3.15}$$

Notice here $Q=\sum_{j}^{m} f_j(s_j\cdot nl)$ is a function of population n. According to (3.9), the F.O.C

for a typical consumer's optimization problem is reduced to

$$\frac{U'(x_{i,t}^{t})}{V'(x_{i,t+1}^{t})}=\frac{U'\left(\frac{r}{p}\cdot l-\frac{1}{n}\sum_{j=1}^{m}\frac{q_{j,t}}{p_t})\right)}{V'\left(\frac{1}{n}\sum_{j=1}^{m}\frac{q_{j,t+1}}{p_{t+1}}+\frac{Q}{n}-\frac{r}{p}\cdot l\right)}$$

$$= \frac{\frac{q_{j,t+1}}{p_{t+1}} + \frac{\pi_{j,t+1}}{p_{t+1}}}{\frac{q_{j,t}}{p_t}}$$

$$= \frac{\frac{q_{j,t+1}}{p_{t+1}} + f_j(s_j L) - \frac{r}{p} \cdot s_j \cdot L}{\frac{q_{j,t}}{p_t}}$$

Notice in the above equation, j is arbitrary, so we have

$$\frac{\frac{q_{i,t+1}}{p_{t+1}} + f_i(s_i L) - \frac{r}{p} \cdot s_i \cdot L}{\frac{q_{i,t}}{p_t}} = \frac{\frac{q_{j,t+1}}{p_{t+1}} + f_j(s_j L) - \frac{r}{p} \cdot s_j \cdot L}{\frac{q_{j,t}}{p_t}}$$

for $\forall i, j = 1,2,...,m.$ Thus we have the following result by summing over j in both numerator and denominator

$$\frac{\frac{q_{j,t+1}}{p_{t+1}} + \frac{\pi_{j,t+1}}{p_{t+1}}}{\frac{q_{j,t}}{p_t}} = \frac{\sum_{j=1}^{m} \frac{q_{j,t+1}}{p_{t+1}} + Q - \frac{r}{p} \cdot L}{\sum_{j=1}^{m} \frac{q_{j,t}}{p_t}}$$

Hence

$$\frac{U'(x_{i,t}^t)}{V'(x_{i,t+1}^t)} = \frac{\frac{1}{n} \sum_{j=1}^{m} \frac{q_{j,t+1}}{p_{t+1}} + \frac{Q}{n} - \frac{r}{p} \cdot l}{\frac{1}{n} \sum_{j=1}^{m} \frac{q_{j,t}}{p_t}} \tag{3.16}$$

We notice from (3.16) that the equation governing the law of motion for the perfectly competitive consumer side has incorporated production, so $\frac{U'(x_{i,t}^t)}{V'(x_{i,t+1}^t)}$ is not simply $\frac{p_t}{p_{t+1}}$. But since consumer side is perfectly competitive, we view our model as a generalization of Grandmont's model (Grandmont (1985)).

It's useful to make the following change of variables. We let

$$\theta_t = \frac{1}{n}\sum_{j=1}^{m}\frac{q_{j,t}}{p_t},$$

$$\widetilde{Q} = \frac{Q}{n},$$

$$\tilde{l} = \frac{r}{p}\cdot l = g(L)\cdot l.$$

After the change of variables, equations (3.7) and (3.8) become

$$x_{i,t}^t = \tilde{l} - \theta_t \tag{3.17}$$

$$x_{i,t+1}^t = \theta_{t+1} + \widetilde{Q} - \tilde{l}\,. \tag{3.18}$$

Also according to money market clearing condition (3.13)

$$\theta_t = \frac{\frac{\overline{M}}{p_t} - Q}{n} \tag{3.19}$$

The input price r_t and output price p_t are functions of money supply $\overline{M}$, aggregate output Q, aggregate input L, number of workers n and a parameter θ_t. Here $\widetilde{Q}$ is the per capita aggregate output, $\tilde{l}$ is the per capita quantity of goods that can be purchased from income (i.e. per capita income in terms of quantity of output goods). The equation

$$\widetilde{Q}-\tilde{l}=\frac{pQ-rL}{pn}$$

implies that $\widetilde{Q}-\tilde{l}$ is the per capita quantity of goods that can be purchased from firms' profit (i.e. per capita profit in terms of the quantity of output goods).

Since $\theta_t=\tilde{l}-x_{i,t}^t(=\frac{1}{n}\sum_{j=1}^{m}\frac{q_{j,t}}{p_t})$, we see that θ_t is the per capita remaining quantity of goods that can be purchased from income after spending on consumption in the first period (i.e. per capita first period savings or investment in terms of quantity of output goods, or per capita second period returns from selling the assets in terms of quantity of output goods). We can also interpret θ_t from the perspective of money supply equation (3.19): it is the per capita difference between the real purchasing power of money and the aggregate output. Notice $0<\theta_t<\tilde{l}$ as asset prices are positive and the consumption in the first period should be positive, and $\tilde{l}\leq\widetilde{Q}$ as per capita profit should be nonnegative. $\widetilde{Q}$ and $\tilde{l}$ are functions of L and independent of any prices (variations of $\widetilde{Q}$ and $\tilde{l}$ can only affect the price ratio $\frac{r}{p}$, but not any particular price), hence a full range of θ_t can be obtained through input price p_t.

Suppose the aggregate output, aggregate input and money supply are all fixed, from equation (3.19) we see that the increase of the output price p_t would lead to the decrease of θ_t, which implies that per capita first period savings or investment in terms of *quantity of* *output goods* would decrease. That is, even if the price ratio $\frac{r}{p}$ is fixed as before, the increase of output price would lead young agents to consume more goods and lead old agents to consume fewer goods. This is perhaps due to the *real income effect*: when the price increases, even though relative price is constant, purchasing power goes up because what the young are selling is worth more.

We also notice from (3.19) that if output price p_t or the aggregate bid $B_t = p_t \cdot Q$ changes in the same proposition to the change of money supply $\overline{M}$, θ_t wouldn't change, given that aggregate output and input are fixed. Hence real variables like consumption wouldn't change.

Given the change of variables, we want to analyze how θ_t evolves over time, i.e. the dynamics of θ_t. The F.O.C for a typical consumer's optimization problem becomes

$$-\theta_t U'(\tilde{l}-\theta_t)+(\theta_{t+1}+\widetilde{Q}-\tilde{l})V'(\theta_{t+1}+\widetilde{Q}-\tilde{l})=0 . \qquad (3.20)$$

Equation (3.20) shows an implicit characterization of the dynamics of our model in terms of state variable θ_t. Before analyzing the dynamics of the model, first we show the existence of steady state equilibria in the model.

Lemma 3.3.1 There exists a steady-state equilibrium.

Proof. We define a function

$$h(\theta)=-\theta U'(\tilde{l}-\theta)+(\theta+\widetilde{Q}-\tilde{l})V'(\theta+\widetilde{Q}-\tilde{l}).$$

Then

$$\lim_{\theta \to 0^+} h(\theta) = (\widetilde{Q} - \tilde{l})V'(\widetilde{Q} - \tilde{l}) \geq 0.$$

The above inequality is strict if and only if $\widetilde{Q} > \tilde{l}$.

First we consider the case when $\widetilde{Q} > \tilde{l}$, we have

$$\lim_{\theta \to 0^+} h(\theta) = (\widetilde{Q} - \tilde{l})V'(\widetilde{Q} - \tilde{l}) > 0.$$

On the other hand, we have

$$\lim_{\theta \to \tilde{l}^-} h(\theta) = -\tilde{l} \lim_{\theta \to \tilde{l}^-} U'(\tilde{l} - \theta) + \widetilde{Q}V'(\widetilde{Q}) < 0$$

according to Inada condition. Since $h(\theta)$ is continuous, it must have a zero in $(0, \tilde{l})$. Second we consider the case when $\widetilde{Q} = \tilde{l}$, we have

$$h(\theta) = \theta(V'(\theta) - U'(\tilde{l} - \theta)).$$

Since $\theta \neq 0$, it suffices to find zeros of function $V'(\theta) - U'(\tilde{l} - \theta)$. Notice

$$\lim_{\theta \to 0^+} V'(\theta) - U'(\tilde{l} - \theta) = \lim_{\theta \to 0^+} V'(\theta) - U'(\tilde{l}) > 0$$

and

$$\lim_{\theta \to \tilde{l}^-} V'(\theta) - U'(\tilde{l} - \theta) = V'(\tilde{l}) - \lim_{\theta \to \tilde{l}^-} U'(\tilde{l} - \theta) < 0$$

according to Inada condition. Since $V'(\theta) - U'(\tilde{l} - \theta)$ is continuous, there must have a zero in

$(0,\tilde{l})$ and the same is for the function $h(\theta)$. In both cases, $h(\theta)$ has a zero in $(0,\tilde{l})$, which implies that there exists a steady-state equilibrium. □

Lemma 3.3.2 The steady state equilibrium $\hat{\theta}$ is a nontrivial function of the average output per worker $\widetilde{Q}$ if $\frac{dh}{d\widetilde{Q}}|_{\theta=\hat{\theta}} \neq 0$.

Proof. Let $\hat{\theta}$ satisfies

$$h(\hat{\theta},\widetilde{Q}) = -\hat{\theta}U'(\tilde{l}-\hat{\theta}) + (\hat{\theta}+\widetilde{Q}-\tilde{l})V'(\hat{\theta}+\widetilde{Q}-\tilde{l}) \equiv 0. \tag{3.21}$$

Notice $\tilde{l} = \frac{r}{p}\cdot l$ is also a function of $\widetilde{Q}$ because $\frac{r}{p}$ is a function of Q according to (3.10).

Differentiating h with respect to $\hat{\theta}$, we get

$$\frac{dh}{d\hat{\theta}} = -U'(\tilde{l}-\hat{\theta}) + V'(\hat{\theta}+\widetilde{Q}-\tilde{l}) + \hat{\theta}U''(\tilde{l}-\hat{\theta}) + (\hat{\theta}+\widetilde{Q}-\tilde{l})V''(\hat{\theta}+\widetilde{Q}-\tilde{l}).$$

Since $h(\hat{\theta},\widetilde{Q}) \equiv 0$, $V' > 0$ and $\widetilde{Q} \geq \tilde{l}$, we have

$$-U'(\tilde{l}-\hat{\theta}) + V'(\hat{\theta}+\widetilde{Q}-\tilde{l}) = \frac{-(\widetilde{Q}-\tilde{l})V'(\hat{\theta}+\widetilde{Q}-\tilde{l})}{\hat{\theta}} \leq 0 \tag{3.22}$$

According to Inada condition $U'' < 0$, $V'' < 0$, together with $\hat{\theta}+\widetilde{Q}-\tilde{l} > 0$ we have

$$\frac{dh}{d\hat{\theta}} < 0.$$

On the other hand, if

$$\frac{dh}{d\widetilde{Q}} = -\hat{\theta}U''(\tilde{l}-\hat{\theta})\cdot\frac{d\tilde{l}}{d\widetilde{Q}} + \left(1-\frac{d\tilde{l}}{d\widetilde{Q}}\right)V'(\hat{\theta}+\widetilde{Q}-\tilde{l}) + (\hat{\theta}+\widetilde{Q}-\tilde{l})\left(1-\frac{d\tilde{l}}{d\widetilde{Q}}\right)V''(\hat{\theta}+\widetilde{Q}-\tilde{l})$$

$$= V'(\hat{\theta}+\widetilde{Q}-\tilde{l}) + (\hat{\theta}+\widetilde{Q}-\tilde{l})V''(\hat{\theta}+\widetilde{Q}-\tilde{l})$$
$$-\frac{d\tilde{l}}{d\widetilde{Q}}\left(\hat{\theta}U''(\tilde{l}-\hat{\theta}) + V'(\hat{\theta}+\widetilde{Q}-\tilde{l}) + (\hat{\theta}+\widetilde{Q}-\tilde{l})V''(\hat{\theta}+\widetilde{Q}-\tilde{l})\right)$$

$$\neq 0$$

and according to $h(\hat{\theta},\widetilde{Q}) \equiv 0$ along with

$$\frac{dh}{d\widetilde{Q}} + \frac{dh}{d\hat{\theta}}\cdot\frac{d\hat{\theta}}{d\widetilde{Q}}\Big|_{\hat{\theta}=\hat{\theta}(\widetilde{Q})} \equiv 0 ,$$

we have

$$\frac{d\hat{\theta}}{d\widetilde{Q}}\Big|_{\hat{\theta}=\hat{\theta}(\widetilde{Q})} = -\frac{\frac{dh}{d\widetilde{Q}}}{\frac{dh}{d\hat{\theta}}} \neq 0,$$

hence $\hat{\theta}$ is a nontrivial function of $\widetilde{Q}$. □

According to the definition of market thickness in Peck et al. (1992), when Q is small/large relative to L, we are tempted to say that the market is thin/thick. Therefore $\frac{\widetilde{Q}}{l} = \frac{Q}{L}$ is a good measure of market thickness. Assuming individual labor offer l is fixed, we use $\widetilde{Q}$ to measure the "thickness" of the market.

We have the following result on market thickness:

Proposition 3.3.1 If $\widetilde{Q}=\tilde{l}$ or

$$\widetilde{Q}\neq\tilde{l}\,,\ \frac{d\hat{\theta}}{d\widetilde{Q}}-\frac{d\tilde{l}}{d\widetilde{Q}}<\frac{\hat{\theta}}{\widetilde{Q}-\tilde{l}}\,,$$

then thick markets are Pareto superior to thin markets.

Proof. Let the lifetime utility function associated with steady-state be

$$W(\hat{\theta},\widetilde{Q})=U(\tilde{l}-\hat{\theta})+V(\hat{\theta}+\widetilde{Q}-\tilde{l})\,.$$

Then

$$\frac{dW}{d\widetilde{Q}}=-U'(\tilde{l}-\hat{\theta})\left(\frac{d\hat{\theta}}{d\widetilde{Q}}-\frac{d\tilde{l}}{d\widetilde{Q}}\right)+V'(\hat{\theta}+\widetilde{Q}-\tilde{l})\left(\frac{d\hat{\theta}}{d\widetilde{Q}}+1-\frac{d\tilde{l}}{d\widetilde{Q}}\right)$$

$$=\left(-U'(\tilde{l}-\hat{\theta})+V'(\hat{\theta}+\widetilde{Q}-\tilde{l})\right)\left(\frac{d\hat{\theta}}{d\widetilde{Q}}-\frac{d\tilde{l}}{d\widetilde{Q}}\right)+V'(\hat{\theta}+\widetilde{Q}-\tilde{l})$$

$$\overset{(3.22)}{=}\frac{-(\widetilde{Q}-\tilde{l})V'(\hat{\theta}+\widetilde{Q}-\tilde{l})}{\hat{\theta}}\cdot\left(\frac{d\hat{\theta}}{d\widetilde{Q}}-\frac{d\tilde{l}}{d\widetilde{Q}}\right)+V'(\hat{\theta}+\widetilde{Q}-\tilde{l})$$

$$=\left(1-\frac{(\widetilde{Q}-\tilde{l})\cdot\left(\frac{d\hat{\theta}}{d\widetilde{Q}}-\frac{d\tilde{l}}{d\widetilde{Q}}\right)}{\hat{\theta}}\right)V'(\hat{\theta}+\widetilde{Q}-\tilde{l})\,.$$

By Inada condition, $V'(\hat{\theta}+\widetilde{Q}-\tilde{l})>0$. If $\widetilde{Q}=\tilde{l}$, then $\frac{dW}{d\widetilde{Q}}>0$. If $\widetilde{Q}\neq\tilde{l}$, since

$\frac{d\hat{\theta}}{d\tilde{Q}} - \frac{d\tilde{l}}{d\tilde{Q}} < \frac{\hat{\theta}}{\tilde{Q} - \tilde{l}}$ is equivalent to $1 - \frac{(\tilde{Q} - \tilde{l}) \cdot \left(\frac{d\hat{\theta}}{d\tilde{Q}} - \frac{d\tilde{l}}{d\tilde{Q}} \right)}{\hat{\theta}} > 0$, we still have $\frac{dW}{d\tilde{Q}} > 0$. Since the steady state marginal utility with respect to average output per worker is strictly positive under the assumption, it follows that utility increases as the market gets thick. □

Notice if $\frac{d\hat{\theta}}{d\tilde{Q}} - \frac{d\tilde{l}}{d\tilde{Q}} = \frac{d(\hat{\theta} - \tilde{l})}{d\tilde{Q}} < 0$, then the assumption in the above proposition naturally holds. Also notice from equation (3.17), $\tilde{l} - \hat{\theta}$ is the steady state per capita first period consumption.

Corollary 3.3.1 If the marginal steady state first period consumption per capita with respect to market thickness is positive, then thick markets are Pareto superior to thin markets.

Proof. Since $\tilde{l} - \hat{\theta}$ is the steady state first period consumption per capita, then if the marginal steady state first period consumption per capita with respect to market thickness is positive, i.e. $\frac{d(\tilde{l} - \hat{\theta})}{d\tilde{Q}} > 0$, then $\frac{d\hat{\theta}}{d\tilde{Q}} - \frac{d\tilde{l}}{d\tilde{Q}} = \frac{d(\hat{\theta} - \tilde{l})}{d\tilde{Q}} < 0 < \frac{\hat{\theta}}{\tilde{Q} - \tilde{l}}$, hence the assumption in Proposition 3.3.1 holds, the conclusion that thick markets are Pareto superior to thin markets holds as well. □

Notice the conditions under which the Pareto rankability of Nash equilibria in terms of market thickness holds is stricter than in the Goenka, et al. (1998) paper.

3.3.1 Backward dynamics

Following Grandmont (1985) and Goenka, et al. (1998), we analyze the backward dynamics of the market game OLG model. The reason for studying backward dynamics instead of forward dynamics is because forward dynamics are not given by a function, but by a correspondence, and hence working with the backward dynamics eliminates the problem. As Grandmont was the first one to note, any stationary equilibrium in the backward dynamics will also be a stationary equilibrium in the forward dynamics. The only difference will be in terms of the stability properties, i.e. the stable state in forward dynamics is the unstable state in backward dynamics.

We write the backward dynamics as $\theta_t = \varphi(\theta_{t+1})$. We further define two functions

$$v_1(\theta) = \theta U'\left(\tilde{l} - \theta\right)$$

and

$$v_2(\theta) = (\theta + \widetilde{Q} - \tilde{l})V'(\theta + \widetilde{Q} - \tilde{l}) .$$

And we have

$$v_1(\varphi(\theta)) = v_2(\theta) .$$

In particular, at steady state,

$$v_1(\hat{\theta}) = v_2(\hat{\theta}) . \qquad (3.23)$$

The first derivatives of $v_1(\theta)$ and $v_2(\theta)$ yield

$$v'_1(\theta) = U'\left(\tilde{l} - \theta\right) - \theta U''\left(\tilde{l} - \theta\right) > 0$$

since $U'>0$ and $U''<0$ and

$$v'_2(\theta) = V'(\theta+\widetilde{Q}-\tilde{l}) + (\theta+\widetilde{Q}-\tilde{l})V''(\theta+\widetilde{Q}-\tilde{l})$$

We see that $v_1(\theta)$ is positive and strictly increasing on the interval $(0,\tilde{l})$. Let

$$R_2(x) = -\frac{V''(x)x}{V'(x)}$$

be the relative risk aversion of the old agent. We make the following assumption on $R_2(x)$:

Assumption 3.3.3 We assume

1. $R_2(x)$ is continuous, positive and strictly increasing on the interval $[\widetilde{Q}-\tilde{l},\widetilde{Q}]$;

2. $R_2(\widetilde{Q}-\tilde{l}) < 1 < R_2(\widetilde{Q})$.

Assumption 3.3.3 implies that $\widetilde{Q}$ has a positive lower bound under assumption. This is because $\widetilde{Q}$ must be greater than $\hat{x}$, which is a positive value satisfying $R_2(\hat{x})=1$, by monotonicity. The assumption that $\widetilde{Q}$ has a positive lower bound is equivalent to the assumption that the number of agents n must satisfy that the average output per capita is bounded away from zero.

Then we have the following results

Proposition 3.3.2 Under Assumption 3.2.1, 3.3.1, 3.3.2, 3.3.3, there exists a unique critical point θ^* of $\varphi(\theta)$ on interval $(0,\tilde{l})$.

Proof. Since

$$v'_2(\theta) = V'(\theta + \widetilde{Q} - \tilde{l})\left(1 + \frac{V''(\hat{\theta} + \widetilde{Q} - \tilde{l})}{V'(\theta + \widetilde{Q} - \tilde{l})} \cdot (\theta + \widetilde{Q} - \tilde{l})\right)$$

$$= V'(\theta + \widetilde{Q} - \tilde{l})\left(1 - R_2(\theta + \widetilde{Q} - \tilde{l})\right),$$

then under Assumption 3.3.3, $\exists \theta^* \in (0, \tilde{l})$ such that $R_2(\theta^* + \widetilde{Q} - \tilde{l}) = 1$ and $v'_2(\theta^*) = 0$. Since $R_2(x)$ is strictly increasing, such θ^* is unique, and $R_2(\theta + \widetilde{Q} - \tilde{l}) < 1$ for $\theta \in (0, \theta^*)$ and $R_2(\theta + \widetilde{Q} - \tilde{l}) > 1$ for $\theta \in (\theta^*, \tilde{l})$. Since $V'(\theta + \widetilde{Q} - \tilde{l}) > 0$ on the interval $(0, \tilde{l})$, we have $v'_2(\theta) > 0$ for $\theta \in (0, \theta^*)$ and $v'_2(\theta) < 0$ for $\theta \in (\theta^*, \tilde{l})$. Notice $v_1(\varphi(\theta)) = v_2(\theta)$, hence $\varphi(\theta) = v_1^{-1}(v_2(\theta))$ and $\varphi'(\theta) = \dfrac{v'_2(\theta)}{v'_1(\varphi(\theta))}$. Since $v_1(\theta)$ is strictly increasing on $(0, \tilde{l})$, it follows that $\varphi(\theta)$ has a unique critical point at θ^* and $\varphi'(\theta) > 0$ for $\theta \in (0, \theta^*)$ and $\varphi'(\theta) < 0$ for $\theta \in (\theta^*, \tilde{l})$. □

Assumption 3.3.3 ensures the uniqueness of critical point θ^*, but this assumption also sets strict restrictions on the choice of utility function $V(x)$. We show in Appendix 3.A how to construct a utility function that satisfies Assumption 3.3.3 and Inada conditions, and we will see that the construction is nontrivial but such utility function exists, which is a combination of preferences with constant relative risk aversions and increasing relative risk aversions. As such utility functions are far more complicated than standard CRRA utility functions showed in Goenka, et al. (1998), we have the hypothesis that for complex dynamics to occur, the choice of utility functions must be very special and the phenomenon of wide dynamics will not be as commonly seen as in similar models without production.

The uniqueness assumption of the critical point saves the trouble of having to restrict the analysis to what happens between the unstable and stable steady states, so in this sense, Assumption 3.3.3 would be necessary to be part of a sufficient condition for complex dynamics to occur. But notice in particular from the proof that for $\varphi'(0) > 0$ to be held true, $R_2(\widetilde{Q} - \tilde{l})$ must be less than 1 and for $\varphi'(\theta) < 0$ to be held true, $R_2(\widetilde{Q} - \tilde{l} + \theta)$ must be greater than 1, $0 < \theta_t < \tilde{l}$. Hence for complex dynamics to occur, at least a modified second part of Assumption 3.3.3 is essential:

$$R_2(\widetilde{Q} - \tilde{l}) < 1 < R_2(\widetilde{Q} - \tilde{l} + \theta), \quad 0 < \theta_t < \tilde{l}. \tag{3.24}$$

We see this modified second part of Assumption 3.3.3 eliminates the possibility of CRRA utility functions to exhibit complex dynamics in our model with production. We will show in Section 3.4 by solving explicitly the backward price dynamics when preferences are log-linear to show the impossibility of the existence of such complex dynamics in our model.

Corollary 3.3.2 There exists a unique interior steady-state equilibrium point $\hat{\theta}$ on interval $(0, \tilde{l})$.

Proof. According to (3.20), we have the following equality

$$-\varphi(\theta_{t+1})U'\left(\tilde{l} - \varphi(\theta_{t+1})\right) + (\theta_{t+1} + \widetilde{Q} - \tilde{l})V'(\theta_{t+1} + \widetilde{Q} - \tilde{l}) = 0.$$

If $\widetilde{Q} > \tilde{l}$, then

$$\varphi(0)\cdot U'(\tilde{l}-\varphi(0))=(\tilde{Q}-\tilde{l})V'(\tilde{Q}-\tilde{l})>0,$$

which implies

$$\varphi(0)>0.$$

If $\tilde{Q}=\tilde{l}$, then

$$\varphi(0)\cdot U'(\tilde{l}-\varphi(0))=0,$$

which implies

$$\varphi(0)=0.$$

We then analyze $\varphi'(0)$ in this case, which is

$$\varphi'(0)=\frac{v'_2(0)}{v'_1(\varphi(0))}=\frac{v'_2(0)}{v'_1(0)}=\frac{V'(0)}{U'(\tilde{l})}>1.$$

From the analysis of $\varphi(\theta)$, which is unimodal on the interval $(0,\tilde{l})$ in Proposition 3.3.2, we see that in both cases there exists a unique interior steady-state equilibrium $\hat{\theta}$ such that $\varphi(\hat{\theta})=\hat{\theta}$. □

3.3.2 Cycles of period 2

We want to analyze the conditions for the existence of cycles of period 2. Cyclic equilibria of order 2 are important because their existence implies the existence of sunspot equilibria. We define $\bar{\theta}=\dfrac{U'(\tilde{l})}{V'(\tilde{Q}-\tilde{l})}$, which is the inter-temporal marginal rate of substitution. We then make the following assumptions:

Assumption 3.3.4 We assume

1. $\bar{\theta} < 1$.

2. $\tilde{Q} > \tilde{l}$.

The first part of Assumption 3.3.4 corresponds to the Samuelson case, where young agents find it optimal to save[31]. According to Goenka, et.al (1998) and Grandmont (1985), under Assumption 3.2.1, 3.3.3, 3.3.4, the sufficient condition for existence of a cycle of period 2 is $\varphi'(\hat{\theta}) < -1$.

Proposition 3.3.3 Under Assumption 3.2.1, 3.3.4, if

$$R_2(\hat{\theta} + \tilde{Q} - \tilde{l}) > \left(2 + \frac{\tilde{Q} - \tilde{l}}{\hat{\theta}}\right) + \left(\frac{\tilde{Q}}{\tilde{l} - \hat{\theta}} - 1\right) \cdot R_1(\tilde{l} - \hat{\theta}),$$

then there exists a cycle of period 2.

Proof. Note at steady state, $\varphi(\hat{\theta}) = \hat{\theta}$. We have

$$\varphi'(\hat{\theta}) = \frac{v'_2(\hat{\theta})}{v'_1(\varphi(\hat{\theta}))}$$

[31] The classical case where $\bar{\theta} > 1$ in a two-period model reduces to autarky, since the classical case involves negative saving to get to the stationary Pareto optimal allocation (as a result, it would require taking resources from the old without ever compensating them for this loss).

$$= \frac{V'(\hat{\theta}+\widetilde{Q}-\tilde{l})}{U'(\tilde{l}-\hat{\theta})} \cdot \frac{1-R_2(\hat{\theta}+\widetilde{Q}-\tilde{l})}{1+\frac{\hat{\theta}}{\tilde{l}-\hat{\theta}} \cdot R_1(\tilde{l}-\hat{\theta})}$$

$$\overset{(3.21)}{=} \frac{\hat{\theta}}{\hat{\theta}+\widetilde{Q}-\tilde{l}} \cdot \frac{1-R_2(\hat{\theta}+\widetilde{Q}-\tilde{l})}{1+\frac{\hat{\theta}}{\tilde{l}-\hat{\theta}} \cdot R_1(\tilde{l}-\hat{\theta})}.$$

Hence $\varphi'(\hat{\theta}) < -1$ is equivalent to

$$R_2(\hat{\theta}+\widetilde{Q}-\tilde{l}) > \left(2+\frac{\widetilde{Q}-\tilde{l}}{\hat{\theta}}\right)+\left(\frac{\widetilde{Q}}{\tilde{l}-\hat{\theta}}-1\right) \cdot R_1(\tilde{l}-\hat{\theta}), \qquad (3.25)$$

i.e. the sufficient condition for the existence of cycles of period 2. □

Notice from Proposition 3.3.3, the condition for the existence of cycles of period 2 is that old agents are sufficiently risk averse. Apparently $R_2(\hat{\theta}+\widetilde{Q}-\tilde{l})$ has to be greater than 2. Under Assumption 3.3.3, $R_2(\widetilde{Q}) > R_2(\hat{\theta}+\widetilde{Q}-\tilde{l}) > 2$, while $R_2(\widetilde{Q}-\tilde{l}) < 1$, so the set of the utility function $V(x)$ such that cycles of period 2 would occur is rather limited.

3.3.3 Cycles of period 3

According to Li and Yorke(1975), if a map ϕ has a cycle of period 3, then it would have a cycle of any period. We aim to apply results of Goenka, et al. (1998) and Grandmont(1985) to find conditions under which a cycle of period 3 occurs. The conditions under which a cycle of period 3 occurs require that for some $\tilde{l}$ and the unique critical point θ^* of φ the following conditions are satisfied

1a. $\varphi\left(\theta^*\right) \le \tilde{l}$ (equivalently, $v_2(\theta^*) \le v_1(\tilde{l})$),

1b. $\exists \theta_0 \ge \theta^*$, s.t. $v_2(\theta^*) > v_1(\theta_0)$,

1c. $v_2(\theta_0) \le v_1\left(\dfrac{U'(\tilde{l})}{V'\left(\tilde{Q}-\tilde{l}\right)} \cdot \theta^*\right)$.

Proposition 3.3.4 Under Assumption 3.3.3, 3.3.4, the following conditions

2a. $\theta^* < \varphi(\theta^*)$,

2b. $\varphi\left(\varphi(\theta^*)\right) < \dfrac{U'(\tilde{l})}{V'\left(\tilde{Q}-\tilde{l}\right)} \cdot \theta^*$,

are equivalent to conditions (1a),(1b) and (1c), where θ^* is the unique critical point of $\varphi(\theta)$.

Proof. We prove the proposition through two directions. First, we want to show conditions (2a) and (2b) imply (1a), (1b) and (1c). Denote $k\left(\theta\right) = v_1\left(\theta\right) - v_2\left(\theta^*\right)$. When $\theta \to 0^+$, $k\left(\theta\right) \to -v_2\left(\theta^*\right) < 0$; when $\theta \to \tilde{l}-$, $k\left(\theta\right) \to +\infty$. Therefore by continuity of $k\left(\theta\right)$ there exists $\hat{\theta}$ such that $k(\hat{\theta}) = 0$. Then $v_1(\hat{\theta}) = v_2\left(\theta^*\right)$ yields $0 < \hat{\theta} = \varphi(\theta^*) < \tilde{l}$, that is, φ maps interval $(0, \tilde{l})$ to itself. This implies condition (1a).

According to Proposition 3.3.2, $\varphi'(\theta) < 0$ for $\theta \in \left(\theta^*, \tilde{l}\right)$. Since $\theta^* < \varphi(\theta^*)$, then $\varphi'(\theta) < 0$ on interval $(\theta^*, \varphi(\theta^*)]$. Under the first part of Assumption 3.3.4, $\theta^* > \dfrac{U'(\tilde{l})}{V'\left(\tilde{Q}-\tilde{l}\right)} \cdot \theta^*$, then $\varphi(\theta^*) > \frac{U'(\tilde{l})}{V'\left(\tilde{Q}-\tilde{l}\right)} \cdot \theta^* > \varphi\left(\varphi(\theta^*)\right)$.

The inverse of $\varphi(\theta)$ may consist of two values, we choose the value that is greater than θ^*. Hence we can have $\theta^* < \varphi^{-1}\left(\frac{U'(\tilde{l})}{V'\left(\tilde{Q}-\tilde{l}\right)} \cdot \theta^* \right) < \varphi(\theta^*)$ as $\varphi'(\theta) < 0$ on interval $(\theta^*, \varphi(\theta^*)]$. We can choose arbitrary θ_0 such that $\varphi^{-1}\left(\frac{U'(\tilde{l})}{V'\left(\tilde{Q}-\tilde{l}\right)} \cdot \theta^* \right) < \theta_0 < \varphi(\theta^*)$ and show that θ_0 satisfies:

1. $\theta_0 \geq \theta^*$, s.t. $v_2(\theta^*) > v_1(\theta_0)$;

2. $v_2(\theta_0) \leq v_1\left(\frac{U'(\tilde{l})}{V'\left(\tilde{Q}-\tilde{l}\right)} \cdot \theta^* \right)$.

First since $\theta^* < \varphi^{-1}\left(\frac{U'(\tilde{l})}{V'\left(\tilde{Q}-\tilde{l}\right)} \cdot \theta^* \right)$ and $\varphi^{-1}\left(\frac{U'(\tilde{l})}{V'\left(\tilde{Q}-\tilde{l}\right)} \cdot \theta^* \right) < \theta_0$, we have $\theta_0 > \theta^*$. Since $\theta_0 < \varphi(\theta^*)$ and $v_1(\theta)$ is strictly increasing, we have $v_1(\theta_0) < v_1\left(\varphi(\theta^*)\right) = v_2(\theta^*)$. Second since $\theta^* < \varphi^{-1}\left(\frac{U'(\tilde{l})}{V'\left(\tilde{Q}-\tilde{l}\right)} \cdot \theta^* \right) < \theta_0 < \varphi(\theta^*)$ and $\varphi'(\theta) < 0$ on interval $(\theta^*, \varphi(\theta^*)]$, we have $\varphi(\theta_0) < \varphi\left(\varphi^{-1}\left(\frac{U'(\tilde{l})}{V'\left(\tilde{Q}-\tilde{l}\right)} \cdot \theta^* \right) \right) = \frac{U'(\tilde{l})}{V'\left(\tilde{Q}-\tilde{l}\right)} \cdot \theta^*$. Again since $v_1(\theta)$ is strictly increasing,

$v_2(\theta_0) = v_1\left(\varphi(\theta_0)\right) \le v_1\left(\frac{U'(\tilde{l})}{V'\left(\tilde{Q}-\tilde{l}\right)} \cdot \theta^*\right)$. Hence both 1&2 are satisfied, which shows conditions (1b) and (1c) are satisfied.

Second, we want to show conditions (1a), (1b) and (1c) imply (2a) and (2b). From (1b), $\exists \theta_0 \ge \theta^*$, s.t. $v_2(\theta^*) > v_1(\theta_0)$. Since v_1 is strictly increasing, v_1^{-1} is also strictly increasing, hence $\varphi(\theta^*) = v_1^{-1} \circ v_2(\theta^*) > \theta_0 \ge \theta^*$, which yields (2a). From (1c), $v_2(\theta_0) \le v_1\left(\frac{U'(\tilde{l})}{V'\left(\tilde{Q}-\tilde{l}\right)} \cdot \theta^*\right)$, since v_1^{-1} is also strictly increasing, we have $\varphi(\theta_0) = v_1^{-1} \circ v_2(\theta_0) \le \frac{U'(\tilde{l})}{V'\left(\tilde{Q}-\tilde{l}\right)} \cdot \theta^*$. Since $\varphi(\theta)$ is decreasing on the interval $\left(\theta^*, \tilde{l}\right)$ and $\theta_0 \in [\theta^*, \varphi(\theta^*))$, we have $\varphi\left(\varphi(\theta^*)\right) < \varphi(\theta_0) \le \frac{U'(\tilde{l})}{V'\left(\tilde{Q}-\tilde{l}\right)} \cdot \theta^*$, hence (2b) is satisfied.

We have proved the equivalence between two sets of conditions. □

Given the form of utility functions, if we want to find out the set of values $\left\{\tilde{Q}, \tilde{l}\right\}$ in which chaos would occur, we consider solving the bifurcation points $\left\{\tilde{Q}, \tilde{l}, \hat{\theta}\right\}$, which are the solutions of the following equations:

$$\frac{U'(\tilde{l})}{V'\left(\tilde{Q}-\tilde{l}\right)} = 1$$

$$\frac{U'(\tilde{l}-\hat{\theta})}{V'\left(\hat{\theta}+\widetilde{Q}-\tilde{l}\right)}=\frac{\hat{\theta}+\widetilde{Q}-\tilde{l}}{\hat{\theta}}$$

$$1+\frac{V''\left(\hat{\theta}+\widetilde{Q}-\tilde{l}\right)}{V'\left(\hat{\theta}+\widetilde{Q}-\tilde{l}\right)}\cdot\left(\hat{\theta}+\widetilde{Q}-\tilde{l}\right)=0$$

(3.26)

The bifurcation points satisfy the conditions in (2a) and (2b) with the exception that inequalities are replaced by equalities, as a result the critical point θ^* and the steady state $\hat{\theta}$ are identical, and the marginal substitution of income is 1. Since there are three equations and three variables, it's possible the bifurcation points are solvable. Analyzing bifurcation points would direct us to find the set of parameters which may satisfy the conditions in (2a) and (2b), in which chaos would occur.

As we see from Assumption 3.3.3, the difficulty with finding the values of $\left\{\widetilde{Q},\tilde{l}\right\}$ lies mainly at specifying the form of utility functions, as the restrictions on relative risk aversion of the utility function are high. We will show in Appendix 3.A how to construct a form of utility function that satisfies Assumption 3.3.3, and empirically, it exists. As a result there is an open set of parameters in which the existence of utility functions that satisfy Assumption 3.3.3 is ensured.

3.4 Special cases: log-linear preferences

We study the price dynamics when utility functions are log-linear. The importance and convenience of log-linearity assumption of preferences on studying overlapping generations models are addressed in Balasko and Shell(1981). Notice log-linear utility functions do not satisfy Assumption 3.3.3. Hence the analysis of price dynamics in section 3.3 does not apply to log-linear utility functions.

We examine the following cases of firms: (1). when $m = 2$; (2). when $m = 3$; (3). when the production functions have the form $f(L) = L$, which is to approximate a pure exchange economy.

(1). $m = 2$:

Without loss of generality, we assume in the following analysis, when the number of firms is 2, the production functions are given by:

Firm 1: $q_1 = f_1(L_1) = A \cdot L_1^2$,

Firm 2: $q_2 = f_2(L_2) = L_2^\alpha$, $0 < \alpha < 1$,

where A and α are constant, f_1 and f_2 are production functions for each firm respectively.

(2). $m = 3$:

Without loss of generality, we assume in the following analysis, when the number of firms is 3, the production functions are given by:

Firm 1: $q_1 = f_1(L_1) = A \cdot L_1^2$,

Firm 2: $q_2 = f_2(L_2) = B \cdot L_2^\alpha$, $0 < \alpha < 1$, Firm 3: $q_3 = f_3\left(L_3\right) = C \cdot L_3$,

where A, B, C and α are constant, f_1, f_2 and f_3 are production functions for each firm respectively.

To make our example more representative, in our example firm 1 has production function of the increasing returns to scale type (IRTS), firm 2 has production function of the decreasing returns to scale type (DRTS), firm 3 has production function of the constant returns to scale type (CRTS).

(3). Constant returns to scale (CRTS) production functions:

When firms have constant returns to scale (CRTS) production functions $f(L) = L$, at time t, firm i's best response is determined by the solution to the optimization problem as follows:

$$\max_{w_{i,t}} \frac{B_t}{Q_t} L_{i,t} - w_{i,t}$$

subject to

$$w_{i,t} \leq \frac{B_t}{Q_t} L_{i,t},$$

and

$$L_{i,t} = \frac{w_{i,t}}{W_t} L_t.$$

The constraint just implies that the net profit should be nonnegative. However, since we all know that each firm has the same production function $f(L) = L$, the aggregate output Q_t is exactly L_t, then the optimization problem setting and solving is a bit different with the arbitrary m firms case. In other words, we can write the optimization problem of firm i as follows:

$$\max_{w_{i,t}} \frac{B_t}{L_t} \frac{w_{i,t}}{W_t} L_t - w_{i,t},$$

equivalently

$$\max_{w_{i,t}} B_t \frac{w_{i,t}}{W_t} - w_{i,t}.$$

The F.O.C with respect to $w_{i,t}$ is

$$\frac{B_t W_{-i,t}}{W_t^2} = 1,$$

where $W_{-i,t} = W_t - w_{i,t}$. Since $p = \frac{B_t}{Q_t} = \frac{B_t}{L_t}$ and $r = \frac{W_t}{L_t}$, we can also write the above equation as

$$\frac{p_t}{r_t} \cdot \frac{W_{-i,t}}{W_t} = 1.$$

Since i is arbitrary, we can conclude that $\frac{w_{i,t}}{W_t} = \frac{1}{m}$ and

$$\frac{p_t}{r_t} = \frac{m}{m-1}.$$

Hence $$\pi_{i,t} = p_t l_{i,t} - r_t l_{i,t} \;= (p_t - r_t)\frac{w_{i,t}}{W_t} L_t$$

$$= (p_t - \frac{m-1}{m} p_t)\frac{1}{m} L_t = \frac{p_t L_t}{m^2}.$$

In the following analysis, we assume $U(x) = \log x$ and $V(x) = \beta \log x$.

3.4.1 Price dynamics

We can show in the appendices that for m firms and the CRTS firms, $\varphi(\theta)$ is a linear function. It is well-known in Li and Yorke's Chaos Theorem (Li and Yorke (1975)) that with the difference equation $x_{t+1} = f\left(x_t\right)$, for chaos to occur, there must exist x in the domain such that $f^3(x) \le x < f(x) < f^2(x)$. Since the price dynamics equations for both m firms and the CRTS firms case are linear, we conclude that neither cycles of period 2 or cycles of period 3 will occur in these cases.

3.4.2 Analysis

We consider two cases when m = 2 and when m = 3. We show the trends of several economic parameters of interest as input labor varies.

3.4.2.1 Two firms case

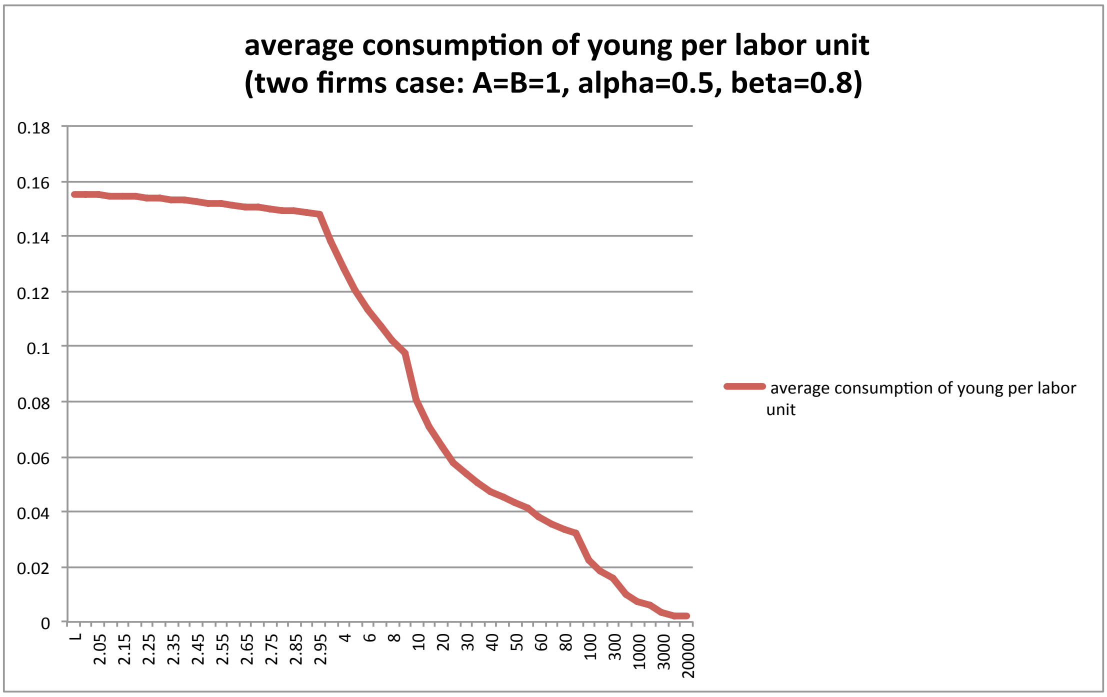
We set the parameters of the two production functions f_1 and f_2 of firm 1 (increasing return to scale) and firm 2(decreasing return to scale) respectively to be $A = B = 1$, $\alpha = 0.5$, $\beta = 0.8$. We do calculations to find average consumption of the young/old, price ratios q_i / r, q_i / p, $i = 1,2$ and r / p.

In the two firms case, as labor increases, the average consumption of the young decreases to almost zero, while the average consumption of the old almost doubles as input good labor doubles according to Figure 3.4.1. The price ratios between two asset prices and input price as labor changes are shown in Figure 3.4.3. For IRTS firm, q_1 / r almost doubles as labor doubles. For DRTS firm, q_2 / r converges to zero. So if input price r is fixed, the asset price for IRTS firm increases in the same speed as the increase of the labor, while the asset price for DRTS firm decreases to almost zero as labor increases. Price ratios r / p and q_i / p, $i = 1,2$, are shown in Figure 3.4.4. We see that r / p converges to zero, q_1 / p increases and q_2 / p converges to zero as labor increases. This shows that if r is fixed, the output price p increases as labor increases and the asset price q_1 for IRTS firm increases faster than output price p.

3.4.2.2 Three firms case

We set the parameters of the three productions function f_1, f_2 and f_3 of firm 1(increasing returns to scale), firm 2 (decreasing returns to scale) and firm 3 (constant returns to scale) respectively to be $A = B = C = 1$, $\alpha = 0.5$, $\beta = 0.8$. We do calculations to find average consumption of the young/old, demand for real balances, price ratios q_i / r, q_i / p, $i = 1,2,3$ and r / p.

In the three firms case, as labor increases, the average consumption of the young increases and converges to some constant as labor increases, while the average consumption of the old almost doubles as labor doubles, as shown by Figure 3.4.2. The price ratios between three asset prices and input price as labor changes are shown in Figure 3.4.3. For IRTS firm, q_1 / r almost doubles as labor doubles. For DRTS firm, q_2 / r converges to zero. For CRTS firm, q_3 / r slowly increases and converges to a constant. So if input price r is fixed, the asset price for IRTS firm increases in the same speed as the increase of labor, the asset price for DRTS firm decreases to almost zero as labor increases and the asset price for CRTS firm converges to almost a constant. Price ratios r / p and q_i / p , $i = 1,2,3$, are shown in Figure 3.4.4. We see that r / p converges to almost a constant. For IRTS firm, q_1 / p almost doubles as labor input doubles. For DRTS firm, q_2 / p converges to zero. For CRTS firm, q_3 / p becomes almost a constant as labor increases. This shows if r is fixed, p will converge to a constant since r / p will converge to a constant. Then q_1 almost doubles as labor input doubles, q_2 converges to zero and q_3 converges to a constant, as labor increases.



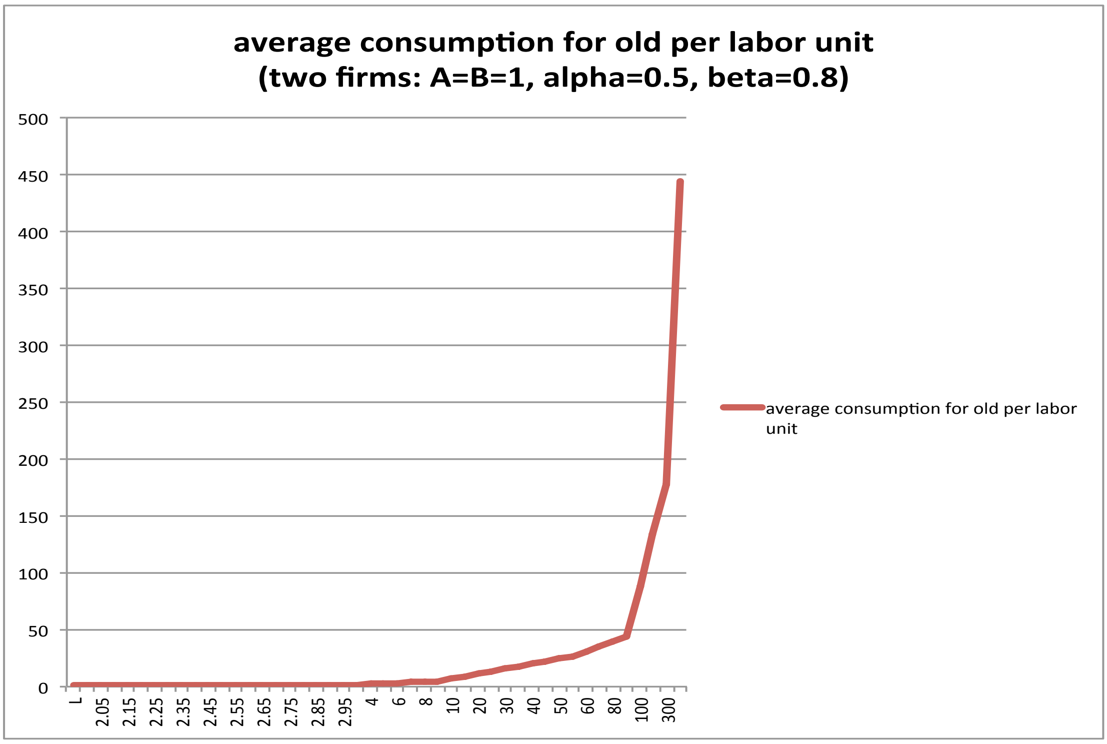
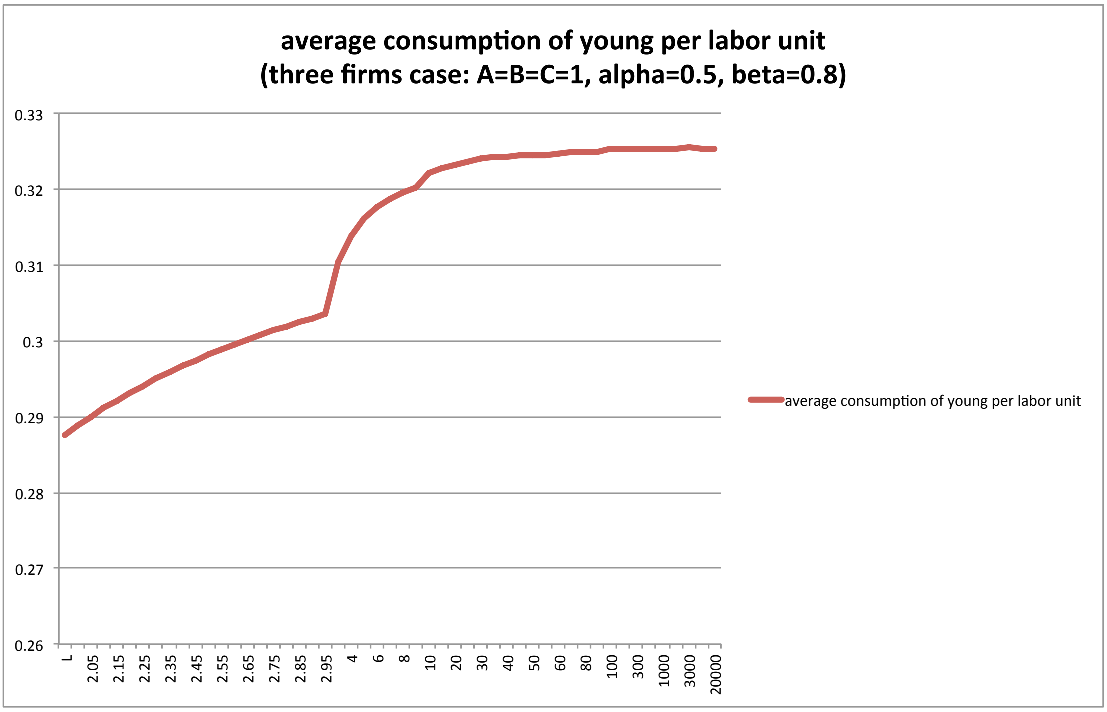


Figure 3.4.1: Average consumption of the young and average consumption of the old in two firms case



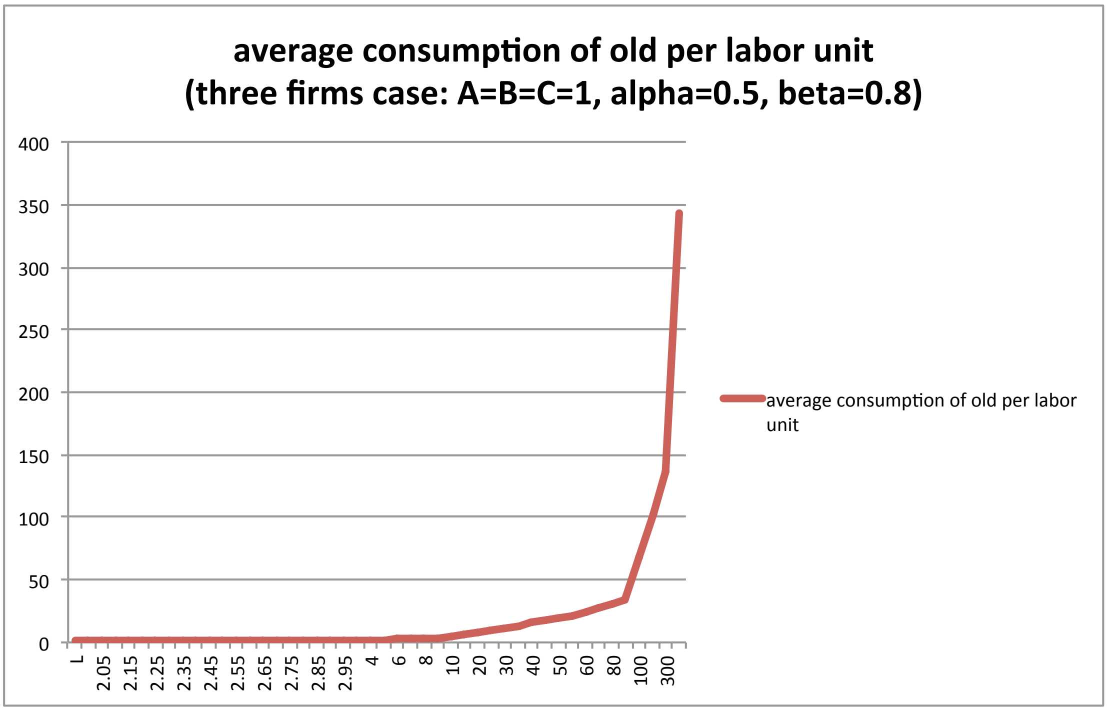
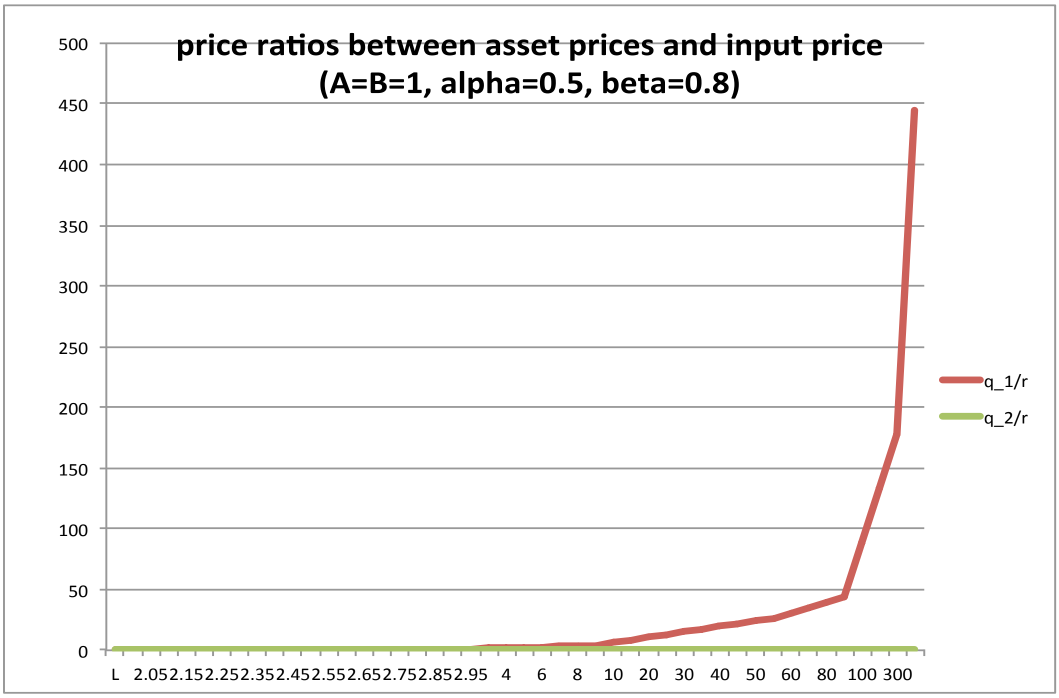


Figure 3.4.2: Average consumption of the young and average consumption of the old in three firms case



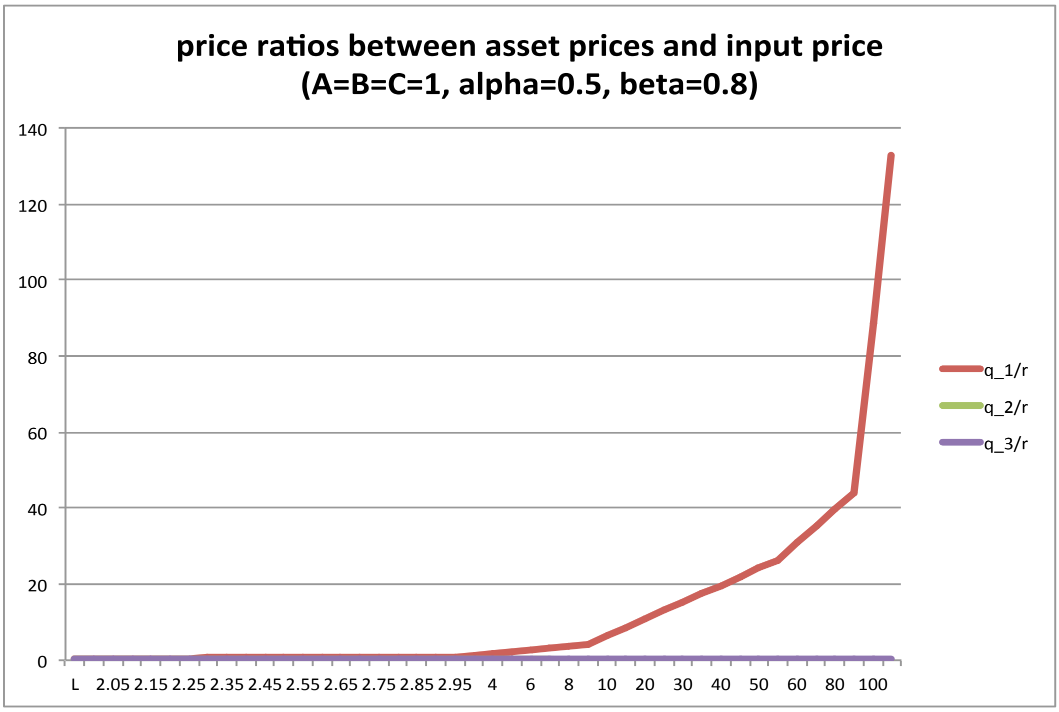


Figure 3.4.3: Price ratios q_i / r, $i = 1,2,$ in two firms case (above) and three firms case (below)

Figure 3.4.4: Price ratios r/p, q_i/r, $i = 1,2,3$, in two firms case (above) and three firms case (below)

3.5 Discussion and conclusions

In this paper, we characterize conditions under which complex and chaotic equilibrium dynamics are possible in the overlapping generations market game model with production, which are more complicated than in the OLG market game model without production. For such complex dynamics to occur, the consumers' preferences in the second period of the overlapping generations model have to be preferences of a mix of relative risk aversions, e.g. a mix of preferences with constant relative risk aversions and increasing relative risk aversions, which are more complicated than in Goenka et al. (1998) or Grandmont (1985). For cycles of period 2 to occur, the old agents must be sufficiently risk-averse, which is similar to the conclusions in Goenka et al. (1998) and the Grandmont (1985). The number of agents must ensure that the average output per worker is bounded away from zero. In general, it must be a very special case for complex dynamics to occur in our model with particular choices of production functions and utility functions. Further, we show the impossibility of such complex dynamics to occur for log-linear preferences, as price dynamics under such preferences are linear.

Our paper extends the Goenka et al. (1998) analysis by adding production, and demonstrates the validity of the conjecture in Goenka et al.'s paper that production would "smooth" out the dynamics, making chaotic dynamics more difficult, and hence less likely to occur. This in turn suggests that there is still need to consider the effects of aggregate shocks, or, in light of Gabbaix's work (2011) (which proposes that a large part of aggregate fluctuations arises from idiosyncratic firm-level shocks if the distribution of firm sizes is fat-tailed), on the effects of sectoral shocks as drivers of business cycles.

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Appendices

Appendix 2.A: Proof of Theorem 2.1

We provide the proof of the generic applicability of the implicit function theorem here. It remains, then, to show that the implicit function theorem (or, more generally, a transversality result) will apply in the neighborhood of the Nash equilibrium for an economy under slack. The Jacobian matrix for the mapping defined by equilibrium conditions has $\Im N + N$ rows (corresponding to the equilibrium first-order conditions and input price equations, respectively), and $(\Im - 1)N + 2N$ columns (corresponding to the input market shares, aggregate input offers, and aggregate expenditures on inputs, respectively). For specificity, we note that we are making a change of variables in the first-order conditions by defining firm k_j's share of aggregate input expenditure on good n as

$$s_{k_j}^n = \frac{w_{k_j}^n}{W^n}.$$

Given this change of variables, variations in the aggregate level of expenditures on inputs holding input expenditure shares constant then means that each firms expenditures scale as the aggregate does. In the input pricing equation

$$r - \widehat{E}^{-1} W = 0 .$$

We take r as a vector of parameters indicating the input price level firms at which would like prices to remain constant. With these definitions, the Jacobian matrix is:

$$\begin{pmatrix} \mathbf{G} & \Phi & \mathbf{0} \\ \mathbf{H} & \Phi_{\Im-1} & \mathbf{0} \\ \mathbf{0} & -\widehat{W}\widehat{E}^{-2} & \widehat{E}^{-1} \end{pmatrix}.$$

The derivatives here are evaluated at the sell-all equilibrium values. The adjustments needed to show the rank result for the short-sale and low employment cases are straight-forward, so we concentrate here on the sell-all game. The derivatives of the first-order conditions with respect to aggregate input expenditures are zero because these always appear in the expenditure share terms, and not alone. The matrix $\mathbf{G}$ is given by

$$\mathbf{G} = \begin{pmatrix} \mathbf{G}_1 & \cdots & \mathbf{0} \\ \vdots & \ddots & \vdots \\ \mathbf{0} & \cdots & \mathbf{G}_{\Im-1} \end{pmatrix},$$

where each matrix $\mathbf{G}_{k_j}$ for $k_j \in \{1,2,...,\Im-1\}$ on the main diagonal is an $N \times N$ matrix given by

$$\mathbf{G}_{k_j} = \left[-p^j \frac{Q^j_{-k_j}}{Q^j} \left[D^2 f_{k_j} - \frac{1}{Q^j} Df_{k_j} Df^T_{k_j} \right] \widehat{E} + \left[\widehat{W}\widehat{W}^{-1}_{-k_j} \right]^2 \hat{r} \right],$$

i.e., the matrix of derivatives of firm first-order conditions with $\widehat{E} = diag\ E$ and $\hat{r} = diag\ r$.

The $(\Im-1)N \times N$ matrix Φ is given by

$$\Phi = \begin{bmatrix} \Phi_1 \\ \vdots \\ \Phi_{\Im-1} \end{bmatrix}$$

and consists of the derivatives with respect to aggregate input offers of the firm first-order conditions, with each submatrix $\mathbf{\Phi}_{k_j}$ given by

$$\mathbf{\Phi}_{k_j} = p^j \frac{Q^j_{k_j}}{Q^j}\left[D^2 f_{k_j} - \frac{1}{Q^j} Df_{k_j} Df^T_{k_j}\right]\left[I - \widehat{W}_{-k_j}\widehat{W}^{-1}\right] + \widehat{W}\widehat{W}^{-1}_{-k_j}\widehat{W}\widehat{E}^{-2}.$$

The matrix $\mathbf{\Phi}_J$ is $N \times N$. The matrix $\mathbf{H}$ is

$$\mathbf{H} = \left[-\mathbf{G}_{\Im} \cdots - \mathbf{G}_{\Im}\right],$$

which reflects the adding up constraint on the input shares.

We note that if production functions are all concave, then each $\mathbf{G}_j$ is positive definite. If some production function f is strictly quasi-concave, then (assuming f is homogeneous of degree δ > 1), the associated derivative matrix $\mathbf{G}$ will be positive definite as long as

$$Df^T\left[D^2 f - \frac{1}{Q} DfDf^T\right]Df = Df^T D^2 fDf - \frac{\|Df\|^2}{Q} Df^T Df = \left[(\delta - 1) - \frac{\|Df\|^2}{Q}\right]\|Df\|^2 < 0,$$

since the strict quasi-concavity assumption implies that the matrix is negative definite in directions orthogonal to Df. This condition, in turn, requires that $\delta < 1 + \frac{\|Df\|^2}{Q}$. In general, though, we can not guarantee definiteness of the $\{\mathbf{G}_j\}$ matrices. We can, however, guarantee that these matrices have full rank generically, and since we will need to make such genericity arguments below, we simply assume this for now.

Now, with each of the $\mathbf{G}_j$ matrices having full rank, we can reduce the Jacobian matrix to the following matrix

$$\begin{pmatrix} \mathbf{G} & \mathbf{0} & \mathbf{0} \\ \mathbf{H} & \Psi & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \widehat{E}^{-1} \end{pmatrix}$$

where $N \times N$ matrix $\Psi = \Phi_3 - \mathbf{HG}^{-1}\Phi$.

If it turns out that the matrix Ψ is singular, then we can perturb the production functions by adding a quadratic quasi-concave perturbation of the form $\varepsilon_{k_j}(\varphi_{k_j} - \overline{\varphi}_{k_j})^T A_{k_j}(\varphi_{k_j} - \overline{\varphi}_{k_j})$ to each firm's production function, where ε_{k_j} is strictly positive and small, $\overline{\varphi}_{k_j}$ is the firm's Nash equilibrium input allocation, and A_{k_j} is an arbitrary bordered negative definite matrix, with bordering vectors colinear with Df_{k_j}. This then allows us to perturb the matrices in $\mathbf{HG}^{-1}\Phi$ (without perturbing the gradients of firm production functions, and hence of Ψ) and guarantee that Ψ has full rank generically.

Appendix 3.A: Construction of utility function

We aim to construct a utility function $u(x)$ such that:

- $u(x)$ satisfies Inada conditions;
- $R(x)$, $x > 0$ is strictly increasing in an interval Η;
- $\exists\ x^*$ such that $x^* \in \mathrm{H}$ and $R(x^*) = 1$.

One guess of the function form for $u(x)$ is

$$u(x) = \frac{x^{1-\theta}}{1-\theta} + \varepsilon e^{-kx},$$

where θ, ε, k are parameters to be fixed ($0 < \theta < 1$, $k > 0$ and ε need not to be small).

3.A.1 Choice of ε

First, we analyze the conditions under which $u(x)$ satisfies Inada conditions. We want

1. $u(x)$ is twice differentiable on (0, ∞),
2. $u'(x) > 0$ and $u''(x) < 0$ for $0 < x < \infty$,
3. $\lim_{x \to 0} u'(x) = +\infty$,
4. $\lim_{x \to +\infty} u'(x) = 0$.

We can always replace $u(x)$ with $u(x) - u(0)$ to satisfy $u(0) = 0$. Note

$$u'(x) = x^{-\theta} - k\varepsilon e^{-kx}$$

$$u''(x) = -\theta x^{-\theta-1} + k^2 \varepsilon e^{-kx}.$$

Then

$$\lim_{x \to 0} u'(x) = \lim_{x \to 0}(x^{-\theta} - k\varepsilon e^{-kx}) = +\infty$$

and

$$\lim_{x\to+\infty} u'(x) = \lim_{x\to+\infty} (x^{-\theta} - k\varepsilon e^{-kx}) = 0\,,$$

hence 1,3,4 of Inada conditions are satisfied. The second condition among Inada conditions is equivalent to

$$k\varepsilon e^{-kx} < x^{-\theta}\,,\ \forall x > 0\,, \tag{3.A.1}$$

$$k^2 \varepsilon e^{-kx} < \theta x^{-\theta-1}\,,\ \forall x > 0\,, \tag{3.A.2}$$

and is thus equivalent to

$$\varepsilon < \min\left\{\frac{x^{-\theta}}{ke^{-kx}}, \frac{\theta x^{-\theta-1}}{k^2 e^{-kx}}\right\}, \forall x > 0\,,$$

which is also equivalent to

$$\varepsilon < \min\left\{\min\left\{\frac{x^{-\theta}}{ke^{-kx}}\right\}, \min\left\{\frac{\theta x^{-\theta-1}}{k^2 e^{-kx}}\right\}\right\}.$$

Next we try to find values for $\min\left\{\frac{x^{-\theta}}{ke^{-kx}}\right\}$ and $\min\left\{\frac{\theta x^{-\theta-1}}{k^2 e^{-kx}}\right\}$.

To find $\min\left\{\frac{x^{-\theta}}{ke^{-kx}}\right\}$, we do some change of variable and let $x = \alpha \cdot \frac{\theta}{k}$, $\alpha > 0$.

Therefore

$$\min\left\{\frac{x^{-\theta}}{ke^{-kx}}\right\} = \frac{1}{k\left(\frac{\theta}{k}\right)^{\theta}} \cdot \min\left\{\left(\frac{e^{\alpha}}{\alpha}\right)^{\theta}\right\},$$

then it suffices to find α such that $\frac{e^{\alpha}}{\alpha}$ is minimized. Let $g(\alpha)=\frac{e^{\alpha}}{\alpha}$, we see that

$$g'(\alpha)=\frac{e^{\alpha}\cdot\alpha-e^{\alpha}}{\alpha^2}=\frac{e^{\alpha}}{\alpha^2}\cdot(\alpha-1).$$

Thus

$$g'(\alpha)<0 \text{ if } \alpha<1,$$

$$g'(\alpha)=0 \text{ if } \alpha=1,$$

$$g'(\alpha)>0 \text{ if } \alpha>1,$$

which implies that $g(\alpha)$ reaches minimum at $\alpha=1$. Hence

$$\min\left\{\frac{x^{-\theta}}{ke^{-kx}}\right\}=\frac{e^{\theta}}{k\left(\frac{\theta}{k}\right)^{\theta}}.$$

Similarly, to find $\min\left\{\frac{\theta x^{-\theta-1}}{k^2e^{-kx}}\right\}$, we again do the change of variable and let $x=\alpha\cdot\frac{\theta}{k}$, $\alpha>0$.

Therefore

$$\min\left\{\frac{\theta x^{-\theta-1}}{k^2e^{-kx}}\right\}=\frac{\theta}{k^2\left(\frac{\theta}{k}\right)^{\theta+1}}\cdot\min\left\{\left(\frac{e^{\alpha}}{\alpha^{\frac{\theta+1}{\theta}}}\right)^{\theta}\right\}.$$

It suffices to find α such that $\frac{e^{\alpha}}{\alpha^{\frac{\theta+1}{\theta}}}$ is minimum. Let $h(\alpha)=\frac{e^{\alpha}}{\alpha^{\frac{\theta+1}{\theta}}}$, then

$$h'(\alpha) = \frac{e^{\alpha}\alpha^{\frac{1}{\theta}}(\alpha - \frac{\theta+1}{\theta})}{\left(\alpha^{\frac{\theta+1}{\theta}}\right)^2}.$$

Then

$$h'(\alpha) < 0 \ \text{ if } \ \alpha < \frac{\theta+1}{\theta},$$

$$h'(\alpha) = 0 \ \text{ if } \ \alpha = \frac{\theta+1}{\theta},$$

$$h'(\alpha) > 0 \ \text{ if } \ \alpha > \frac{\theta+1}{\theta},$$

which implies that $h(\alpha)$ is minimized at $\alpha = \frac{\theta+1}{\theta}$. Hence

$$\min\left\{\frac{\theta x^{-\theta-1}}{k^2 e^{-kx}}\right\} = \frac{\theta e^{\theta+1}}{k^2\left(\frac{\theta+1}{k}\right)^{\theta+1}}.$$

In all, if ε satisfies

$$\varepsilon < \min\left\{\frac{e^{\theta}}{k\left(\frac{\theta}{k}\right)^{\theta}}, \frac{\theta e^{\theta+1}}{k^2\left(\frac{\theta+1}{k}\right)^{\theta+1}}\right\}, \qquad (3.A.3)$$

then the second condition of Inada conditions is satisfied.

3.A.2 x^*: $R(x^*)=1$

Given $u(x)$, the relative risk aversion is

$$R(x)=-\frac{u''(x)x}{u'(x)}=\frac{\theta x^{-\theta}-k^2\varepsilon\cdot xe^{-kx}}{x^{-\theta}-k\varepsilon\cdot e^{-kx}}.$$

Let x^* be the point that satisfies $R(x^*)=1$. Then x^* satisfies

$$k\varepsilon\cdot e^{-kx}(1-kx)=(1-\theta)x^{-\theta}. \tag{3.A.4}$$

According to equation (3.A.1), we have

$$(1-\theta)x^{*-\theta}<x^{*-\theta}(1-kx^{*-\theta}),$$

hence x^* satisfies

$$x^*<\frac{\theta}{k}. \tag{3.A.5}$$

3.A.3 H: the interval in which $R'(x^*)>0$

That $R'(x)>0$ is equivalent to

$$(\theta x^{-\theta}-k^2\varepsilon\cdot xe^{-kx})'\cdot(x^{-\theta}-k\varepsilon\cdot e^{-kx})-(x^{-\theta}-k\varepsilon\cdot e^{-kx})'(\theta x^{-\theta}-k^2\varepsilon\cdot xe^{-kx})>0,$$

which is equivalent to

$$kx+\frac{\theta^2}{kx}+k\varepsilon\cdot x^{\theta}e^{-kx}-2\theta-1>0\cdot \qquad (3.A.6)$$

Let $f(x)=kx+\frac{\theta^2}{kx}+k\varepsilon\cdot x^{\theta}e^{-kx}-2\theta-1$. Notice when $x=\frac{\theta}{k}$, $f(\frac{\theta}{k})=k\varepsilon\cdot(\frac{\theta}{k})^{\theta}e^{-\theta}-1$.

According to equation (3.A.3), $\varepsilon<\frac{e^{\theta}}{k(\frac{\theta}{k})^{\theta}}$, hence $f(\frac{\theta}{k})<0$. Also note $\lim_{x\to 0}f(x)=+\infty\cdot$

We aim to find an interval in which $R'(x^*)>0$, then by continuity of $R'(x)$, there is an interval containing x^* such that $R(x)$ is strictly increasing on this interval. To find such conditions, first x^* must satisfy equation (3.A.6), i.e. $f(x^*)>0$. Second, we want to locate the lower and upper bound of the targeted interval, which are two points $\underline{x}$ and $\overline{x}$ that are nearest to x^* and satisfy $f(x)=0$ or $x=0$. Both lower bound $\underline{x}$ and upper bound $\overline{x}$ exist: the upper bound is greater than x^* and less than $\frac{\theta}{k}$ because $f(x)$ changes signs between x^* and $\frac{\theta}{k}$; since $\lim_{x\to 0}f(x)=+\infty$ and $f(x^*)>0$, the lower bound is either 0 or the largest zero of $f(x)$ less than x^*.

We can still do some change of variables to make things simpler. Let $\overline{x}=\beta_1\cdot\frac{\theta}{k}$ and $x^*=\beta_2\cdot\frac{\theta}{k}$. Thus equation (3.A.5) is equivalent to $\beta_2<1$. That $\overline{x}$ satisfies $R'(\overline{x})=0$, i.e. $f(\overline{x})=0$ is equivalent to

$$(\beta_1 + \frac{1}{\beta_1} - 2)\theta + k\varepsilon \cdot \left(\beta_1 \frac{\theta}{k}\right)^{\theta} e^{-\beta_1\theta} - 1 = 0. \qquad (3.A.7)$$

And $R'(x^*) > 0$ is equivalent to

$$(\beta_2 + \frac{1}{\beta_2} - 2)\theta + k\varepsilon \cdot \left(\beta_2 \frac{\theta}{k}\right)^{\theta} e^{-\beta_2\theta} > 1. \qquad (3.A.8)$$

Equation (3.A.4) is equivalent to

$$k\varepsilon \cdot \left(\beta_2 \frac{\theta}{k}\right)^{\theta} e^{-\beta_2\theta} = \frac{1-\theta}{1-\beta_2\theta}. \qquad (3.A.9)$$

Substituting (3.A.9) into (3.A.8), we have that $R'(x^*) > 0$ is equivalent to

$$(\beta_2 + \frac{1}{\beta_2} - 2)\theta + \frac{1-\theta}{1-\beta_2\theta} > 1. \qquad (3.A.10)$$

Let $q(x) = \left(x + \frac{1}{x} - 2\right)\theta + \frac{1-\theta}{1-\theta x}$. $\forall\theta, \lim_{x\to 0} q(x) = +\infty$, hence given θ, $\{x : q(x) > 1\}$ is a nonempty open set.

Since $x^* < \bar{x}$ and $f(1 \cdot \frac{\theta}{k}) < 0$, we have that

$$\beta_2 < \beta_1 < 1. \qquad (3.A.11)$$

3.A.4 Cycles of period 2

In the main article, from equation (3.25) we know for cycles of period 2 to occur, it must be that $R(\bar{x}) > 2$. By substituting $\bar{x}$ with $\beta_1 \cdot \frac{\theta}{k}$, we have that $R(\bar{x}) > 2$ is equivalent to

$$k\varepsilon \cdot \left(\beta_2 \frac{\theta}{k}\right)^{\theta} e^{-\beta_1\theta} > \frac{2-\theta}{2-\beta_1\theta}. \tag{3.A.12}$$

By substituting according to equation (3.A.7), we have further that $R(\bar{x}) > 2$ is equivalent to

$$(\beta_1 + \frac{1}{\beta_1} - 2)\theta + \frac{2-\theta}{2-\beta_1\theta} < 1. \tag{3.A.13}$$

Let $p(x) = \left(x + \frac{1}{x} - 2\right)\theta + \frac{2-\theta}{2-\theta x}$. We can prove that $\forall \theta < 1$, $\{x : p(x) < 1\}$ is nonempty.

Consider $p'(x) = \left(1 - \frac{1}{x^2}\right)\theta + \frac{\theta(2-\theta)}{(2-\theta x)^2}$. We see that $\lim_{x \to 0} p'(x) = -\infty$ and

$\lim_{x \to 1} p'(x) = \frac{\theta}{2-\theta} > 0$. Then the critical point of $p(x)$ is within interval (0,1) and hence

$\min_{x \in (0,1)} p(x) < p(1) = 1$, which implies that $\exists x'$ such that $p(x') < 1$. By continuity of $p(x)$,

given θ, $\{x : p(x) < 1\}$ is a nonempty open set.

3.A.5 Summary

Our method is to try many values of k, θ, ε and keep the right ones by checking whether the resulting utility function satisfies the desired conditions according to the following procedure:

Step 1: Select k, θ and ε such that (3.A.3) holds.

Step 2: Solve β_2 from (3.A.9) (note there may be multiple roots). Keep those roots if $\beta_2 < 1$ and continue. If not, return to Step 1.

Step 3: Check if (3.A.10) holds. If yes, we find a critical point $x^* = \beta_2 \cdot \frac{\theta}{k}$ such that $R'(x^*) > 0$. If not, return to Step 1.

Step 4: Solve β_1 from (3.A.7) (note there may be multiple roots), choose the ones which are nearest to (greater than or less than) β_2 and less than 1. If $\beta_1 > \beta_2$, then the upper bound for H is $\bar{x} = \beta_1 \cdot \frac{\theta}{k}$. If $\beta_1 < \beta_2$, then the lower bound for H is $\underline{x} = \beta_1 \cdot \frac{\theta}{k}$. If we only have one value for β_1 which is greater than β_2, then the lower bound for H is 0.

Step 5: We can further examine the possibility of the occurrence of cycles of period 2 by checking if (3.A.13) holds. If not, then there's no possibility of cycles of period 2.

Numerical results show that there exist k, θ, ε such that $R(x)$ is strictly increasing in H. Thus we find utility functions that satisfy the three conditions at the beginning of the appendix. The problem of finding right values for parameters of the proposed utility function $u(x)$ for the possibility of cycles of period 2 to occur is reduced to solving five variables k, θ, ε, β_1, β_2 from two equations (3.A.7) and (3.A.9), two inequalities (3.A.10) and (3.A.13), and constraints (3.A.3), (3.A.11) and $k > 0$, $0 < \theta < 1$.

3.A.6 Numerical results

The following table shows the parameter values such that $R(x)$ is strictly increasing in H and the lower and upper bounds of H:

Table 3.A.1: Parameter values k, θ, ε, β and x^*

k	θ	ε	β	x^*	Lower bound	Upper bound
5	0.8	0.81	0.2432	0.04	0	0.066
10	0.8	0.71	0.2387	0.02	0	0.0331
0.5	0.8	1.29	0.2400	0.38	0	0.6609
5	0.8	1	0.1536	0.02	0	0.0699
10	0.8	1	0.1203	0.01	0	0.0368
0.5	0.8	1.5	0.1707	0.27	0	0.6873
0.1	0.8	2	0.1829	1.46	0	3.4032
0.2	0.8	2	0.1404	0.56	0	1.7774
0.01	0.8	3	0.2061	16.49	0	33.5387
0.01	0.8	4	0.1194	9.55	0	36.8793

Appendix 3.B: log-linear preferences

3.B.1 Price dynamics for arbitrary *m* firms

According to (3.9), we have

$$\frac{x_{i,t+1}^{t}}{x_{i,t}^{t}} = \beta\frac{q_{1,t+1}+\pi_{1,t+1}}{q_{1,t}}\cdot\frac{p_t}{p_{t+1}} \tag{3.B.1}$$

$$= \beta\frac{q_{j,t+1}+\pi_{j,t+1}}{q_{j,t}}\cdot\frac{p_t}{p_{t+1}}, \quad j=1,2,\dots,m. \tag{3.B.2}$$

Hence
$$p_{t+1}x_{i,t+1}^{t}=\frac{x_{i,t+1}^{t}}{x_{i,t}^{t}}\cdot\frac{p_{t+1}}{p_{t}}\cdot p_{t}x_{i,t}^{t}$$

$$\overset{(3.B.2)}{=}\beta\frac{q_{1,t+1}+\pi_{1,t+1}}{q_{1,t}}\cdot\frac{p_{t}}{p_{t+1}}\cdot\frac{p_{t+1}}{p_{t}}\cdot p_{t}x_{i,t}^{t}$$

$$=\beta\frac{q_{1,t+1}+\pi_{1,t+1}}{q_{1,t}}\cdot p_{t}x_{i,t}^{t}$$

$$\overset{(3.7)}{=}\beta\frac{q_{1,t+1}+\pi_{1,t+1}}{q_{1,t}}\left(r_{t}l_{i,t}-\sum_{j=1}^{m}a_{i,t}^{j}q_{j,t}\right)$$

$$\overset{(3.8)}{=}\sum_{j=1}^{m}(q_{j,t+1}+\pi_{j,t+1})a_{i,t}^{j}$$

$$\overset{(3.B.2)}{=}\frac{q_{1,t+1}+\pi_{1,t+1}}{q_{1,t}}\cdot\sum_{j=1}^{m}a_{i,t}^{j}q_{j,t}.$$

Hence

$$\beta\cdot\left(r_{t}l_{i,t}-\sum_{j=1}^{m}a_{i,t}^{j}q_{j,t}\right)=\sum_{j=1}^{m}a_{i,t}^{j}q_{j,t}.$$

We have

$$x_{i,t}^{t}=\frac{r_{t}}{p_{t}}\cdot\frac{l_{i,t}}{1+\beta}$$

$$x_{i,t+1}^{t}=\frac{\beta}{1+\beta}\cdot\frac{q_{1,t+1}+\pi_{1,t+1}}{q_{1,t}}\cdot\frac{r_{t}}{p_{t+1}}\cdot l_{i,t}.$$

Hence

$$x_{i,t}^{t-1} = \frac{\beta}{1+\beta} \cdot \frac{q_{1,t} + \pi_{1,t}}{q_{1,t-1}} \cdot \frac{r_{t-1}}{p_t} \cdot l_{i,t-1}.$$

We assume at each period labor L_t is constant, i.e. $L_t = L$ and $\frac{r_t}{p_t} = \frac{r_{t-1}}{p_{t-1}} = g(L)$. The money market clearing condition becomes

$$\begin{aligned}
\sum_{i=1}^{n} (x_{i,t}^{t} + x_{i,t}^{t-1}) &= \frac{r_t}{p_t} \cdot \frac{1}{1+\beta} \cdot \sum_{i=1}^{n} l_{i,t} + \frac{\beta}{1+\beta} \cdot \frac{q_{1,t} + \pi_{1,t}}{q_{1,t-1}} \cdot \frac{r_{t-1}}{p_t} \cdot \sum_{i=1}^{n} l_{i,t-1} \\
&= \left(\frac{r_t}{p_t} \cdot \frac{1}{1+\beta} + \frac{\beta}{1+\beta} \cdot \frac{q_{1,t} + \pi_{1,t}}{q_{1,t-1}} \cdot \frac{r_{t-1}}{p_t} \right) \cdot L \\
&= \left(\frac{r_t}{p_t} \cdot \frac{1}{1+\beta} + \frac{\beta}{1+\beta} \cdot \frac{q_{1,t} + \pi_{1,t}}{q_{1,t-1}} \cdot \frac{r_{t-1}}{p_{t-1}} \cdot \frac{p_{t-1}}{p_t} \right) \cdot L \\
&= \frac{r}{p} \cdot \frac{1}{1+\beta} \cdot \left(1 + \beta \left(\frac{q_{1,t} + \pi_{1,t}}{q_{1,t-1}} \cdot \frac{p_{t-1}}{p_t} \right) \right) \cdot L \\
&= g(L) \cdot L \cdot \frac{1}{1+\beta} \cdot \left(1 + \beta \left(\frac{q_{1,t} + \pi_{1,t}}{q_{1,t-1}} \cdot \frac{p_{t-1}}{p_t} \right) \right) \\
&= Q
\end{aligned}$$

Hence

$$\frac{p_{t-1}}{p_t} \cdot \frac{q_{1,t} + \pi_{1,t}}{q_{1,t-1}} = \frac{\beta + 1}{\beta} \cdot \frac{Q}{g(L) \cdot L} - \frac{1}{\beta}. \qquad (3.B.3)$$

Denote $\frac{q_{i,t}}{p_t} = \tilde{q}_{i,t}$, $i = 1,2,3$. Notice

$$
\begin{aligned}
\frac{p_{t-1}}{p_t}\cdot\frac{q_{1,t}+\pi_{1,t}}{q_{1,t-1}} &= \frac{\dfrac{q_{1,t}}{p_t}+\dfrac{\pi_{1,t}}{p_t}}{\dfrac{q_{1,t-1}}{p_{t-1}}} \\
&= \frac{\tilde{q}_{1,t}+\dfrac{p_t\cdot f_1(s_1L)-r_ts_1L}{p_t}}{\tilde{q}_{1,t-1}} \\
&= \frac{\tilde{q}_{1,t}+f_1(s_1L)-\dfrac{r_ts_1L}{p_t}}{\tilde{q}_{1,t-1}} \\
&= \frac{n\theta_t+Q-g(L)\cdot L}{n\theta_{t-1}} \\
&= \frac{\theta_t+\tilde{Q}-\tilde{l}}{\theta_{t-1}}.
\end{aligned}
$$

According to (3.B.3), notice $\tilde{Q}$ and $\tilde{l}$ are functions of L,

$$
\theta_t = \frac{\theta_{t+1}+\tilde{Q}-\tilde{l}}{\dfrac{\beta+1}{\beta}\cdot\dfrac{Q}{g(L)\cdot L}-\dfrac{1}{\beta}}, \tag{3.B.4}
$$

which shows price dynamics is linear.

3.B.2 Special case: CRTS firms

We study price dynamics like in the previous section. Without loss of generality, we consider the case when utility function is log linear. The money market clearing condition becomes

$$\sum_{i=1}^{n}\left(x_{i,t}^{t}+x_{i,t}^{t-1}\right)=\frac{r}{p}\cdot\frac{1}{1+\beta}\cdot\left(1+\beta\cdot\frac{p_{t-1}}{p_t}\cdot\frac{q_{1,t}+\pi_{1,t}}{q_{1,t-1}}\right)\cdot L$$

$$=\frac{m-1}{m}\cdot\frac{1}{1+\beta}\left(1+\beta\cdot\frac{p_{t-1}}{p_t}\cdot\frac{q_{1,t}+\frac{p_t L_t}{m^2}}{q_{1,t-1}}\right)\cdot L$$

$$=L$$

Denote $\frac{q_{i,t}}{p_t}=\tilde{q}_t,\ i=1,2,...,m,$ since all asset prices are identical. Therefore

$$\frac{\tilde{q}_t+\frac{L_t}{m^2}}{\tilde{q}_{t-1}}=\frac{p_{t-1}}{p_t}\cdot\frac{q_{1,t}+\frac{p_t L_t}{m^2}}{q_{1,t-1}}=\frac{m}{m-1}\cdot\frac{1+\beta}{\beta}-\frac{1}{\beta}=\frac{n\theta_t+\frac{L_t}{m}}{n\theta_{t-1}},$$

and hence

$$\theta_t=\frac{\theta_{t+1}+\frac{l}{m}}{\frac{m}{m-1}\cdot\frac{1+\beta}{\beta}-\frac{1}{\beta}}, \tag{3.B.5}$$

which shows price dynamics is linear and independent of L.