Three Essays on Extensive-form Games and Strategic Complements

By

Yue Feng

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Chair: Dr. Tarun Sabarwal

Dr. Bernard Cornet

Dr. Zongwu Cai

Dr. Jianbo Zhang

Dr. Rodolfo H. Torres

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The dissertation committee for Yue Feng

certifies that this is the approved version of the following dissertation:

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Chair: Dr. Tarun Sabarwal

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Abstract:

Games with strategic complements have the property that best response set of a player is increasing in the strategies of opponents in the standard lattice set order. Normal form games with strategic complements have been extensively studied in the literature. Such property in extensive form games are less well studied and turns out to be hard to analyze. This dissertation studies strategic complements in extensive form games.

Echenique (2004) has a definition for extensive form game with strategic complements. We show in Chapter 2 that there are many cases of interest that are beyond his definition. Even simple two-stage 2×2 game does not satisfy his definition. More generally, we characterize when two-stage 2×2 game have strategic complements. This shows the aspects of strategic complements not captured by standard definition.

Another way to analyze 2-stage games is to look at their reduced normal form (Mailath, Samuelson and Swinkels (1993)). In Chapter 3, we show that standard ordinal strategic complementarity conditions imposed on the reduced normal form are not sufficient to generate strategic complements in the extensive form. Semi no crossing conditions are added to make it sufficient. Moreover, I provide conditions on the reduced normal form to characterize strategic complements in the two-stage games. I also show how to recover the extensive form from the reduced normal form by applying the algorithm in (MSS (1994)).

In Chapter 4, I study strategic complements in more general two-player multi-stage games. I show that in response to different classes of opponent's strategies, the corresponding best response sets must include strategies that can generate certain paths of play once the conditions are met. I then characterize the structure of best response sets necessary to exhibit strategic complements.

Contents

Chapter 1

1 Introduction

Games of strategic complements (GSC) and Games of strategic substitutes (GSS) formalize basic strategic interactions and have widespread applications. In GSC, best response of each player is weakly increasing in actions of the other players, whereas GSS have the characteristic that the best response of each player is weakly decreasing in the actions of the other players.

This paper focused on games with strategic complements. A few key papers laid the theoretical foundation for this line of research. Supermodular games were first introduced by Topiks (1979), followed by Bulow, Geanakoplos, and Klemperer (1985), Lippman, Mamer, and McCardle (1987), Sobel (1988). And further analyzed by Vives (1990) and Milgrom and Roberts (1990) in which they slightly weakened the definition of supermodular games, contributed to new examples and extended the analysis to broader equilibrium concepts. Milgrom and Shannon (1994) provided ordinal notions of complements and thus defining a more generalized class of games with complementarity property. In this body of work, static games with complements have been investigated. Strong results on the existence of Nash equilibria have been derived, along with comparative statics theorems. Along with Milgrom and Roberts (1994), Zhou (1994), Shannon (1995), Villas-Boas (1997), Edlin and Shannon (1998), Echenique (2002), Echenique (2004) and Quah (2007), among others. Extensive bibliographies are available in Topkis (1998), in Vives (1999), and in Vives (2005).

The toolbox commonly used to deal with complements are monotone comparative statics. This technique is used to exploit fully both order and monotonicity properties. Two critical concepts were introduced, that is, supermodularity and increasing difference. In Euclidean applications, supermodularity means that increasing any subset of the decision variable raises the incremental returns associated with increases in the others. Similarly, increasing differences means that increasing a parameter raises the marginal return to activities. As Topkis (1978) indicates, supermodularity and increasing differences are easily characterized for smooth functions on $\mathbb{R}^n$. Quasi-supermodularity conditions and single crossing conditions used in Milgrom and Shannon (1994) are weaker than the two cardinal notions. In particular, supermodular functions are important subset of quasisupermodular functions. The actual relation between supermodularity and quasisupermodularity involves transformations on restricted domains.

Games with strategic substitutes (GSS) in which the payoff function satisfying decreasing single-crossing condition cannot always be considered as a GSC. Roy and Sabarwal (2012) used a three-player Hawk Dove game to demonstrate the standard technique of reversing the order on strategy space of a play in a GSS to yield a GSC failed to apply to more than one player.

This class of games have deeper connection with strategic situations such as arms races, banks runs, currency crisis, Research and development race, technology adoption and many other topics (Milgrom and Roberts (1990), Topkis (1998), and Vives (1999)). Thus normal form games with strategic complements are considered to be very common and useful game structure.

However, researches has been done but not exhaustively explored beyond the scope of static games and into dynamic games with strategic complements.

Within the scope of dynamic games with strategic complements, two directions have been explored, that is, extensive form games (Echenique(2004)) and stochastic games (Amir (1996, 2010), Curtat (1993), Balbus, Reffette and Wozny (2014)).

Echenique (2004) extended the static concept into extensive form games and defined extensive form game of strategic complementarities, that is, all normal form of the subgames in the extensive form are normal form games with strategic complementarities respecting Milgrom and Shannon's definition. He showed that the set of subgame-perfect equilibria in this class of games is a non-empty, complete lattice. The result is limited because extensive-form games of strategic complementarities defined in this way turned out to be a very restrictive class of games.

Full characterization of the set of equilibria of a stochastic game is an intractable problem in many cases, due to the complex dynamic structure of these equilibria. The literature choose to look at a relatively tractable objective, Markov equilibria. In a Markov equilibria, player's action at each period are a function of current state variable only. Even those equilibria with relatively simple dynamic structure are difficult to analyze.

Curtat (1993) proved the existence of stationary Markov equilibria in pure strategies for a class of stochastic games with assumption of complementarity, monotonicity and diagonal domiance. In Amir(2001), with one-period rewards and state transitions satisfying some complementarity and monotonicity conditions, the existence of Markov-stationary equilibria is proved for the finite and infinite horizon game. Balbus, Reffette and Wozny (2014) studied infinite discounted stochastic games with strategic complementarities and proved the existence of a stationary Markov Nash equilibrium as well as provided method for constructing this least and greatest equilibrium via a simple approximation schemes.

I focused on extensive form games and strategic complements, in particular, an important class of extensive form games: multi-stage games.

In Chapter 2, we study a simple two-stage 2×2 game. Under differential payoff to outcome assumption, we first characterize when a player exhibits strategic complements in this game. Echenique (2004)'s definition for extensive form game with strategic complements, an extension of standard ordinal complementarities conditions in normal form games, rules out all such cases. Moreover, we show when a player exhibits strategic complements in a general two-stage 2×2 game removing the payoff restrictions. This shows the aspects of strategic complements not captured by standard definition.

For extensive form games, the dimensions of strategy space can be huge. This imposed considerable difficulty in analysing strategic complements. In Chapter 3, we show that it is possible to properly reduce the strategy space (MSS(1994)) to facilitate the process without loosing generality in a two-stage 2×2 game. we show that standard ordinal strategic complementarity conditions imposed on the reduced normal form are not sufficient to generate strategic complements in the extensive form. Semi no crossing conditions are added to make it sufficient. Moreoever, I provide conditions on the reduced normal form to characterize strategic complements in the two-stage games.

In Chapter 4, I study the strategic complements in general multi-stage 2-player games. I show that in response to different classes of opponent's strategies, to generate strategic complements, the corresponding best response sets must include all strategies that generate certain paths of play once the conditions are met. I characterize the structure of best response sets necessary to exhibit strategic complements.

Chapter 2

2 Two-stage Games with Strategic Complements

2.1 Example

We start with a motivating two-stage 2×2 game example. We show that this simple extensive form game exhibits strategic complements. We also point out that it is ruled out by the existing definition of extensive form game of strategic complements.

It is commonly accepted that a game exhibits strategic complements if player i's best response sets weakly increase in the standard lattice set order when player i's opponents increases their strategies. Consider the following two-stage 2×2 game with the assigned payoffs. In the first period, a 2×2 game is played with the normal form in Figure 1.

	B_1^0	B_2^0
A_1^0	3, 3	1, -2
A_2^0	-2, 1	-4, -4

Figure 1: *Stage* 1 *game*

Depending on the outcome in the first period, a second stage game is played with the normal forms in Figure 2. In particular, game 1 is played if A_1^0 and B_1^0 are played in the first stage, game 2 is played if A_1^0 and B_2^0 are played in the first stage. Following a play of A_2^0 and B_1^0, game 3 is played. And if A_2^0 and B_2^0 is played, then game 4 is played in the second stage.

	B_1^1	B_2^1
A_1^1	15, 15	10, 5
A_2^1	5, 10	0, 0

	B_1^2	B_2^2
A_1^2	5, 5	0, 15
A_2^2	15, 0	10, 10

	B_1^3	B_2^3
A_1^3	0, 0	5, 15
A_2^3	15, 5	10, 10

	B_1^4	B_2^4
A_1^4	15, 15	5, 0
A_2^4	0, 5	10, 10

Figure 2: *Stage 2 games*

Figure 3. is the extensive form of the two stage game with payoff assigned at terminal nodes. We assume the discount factor is $\delta = 0.8$.

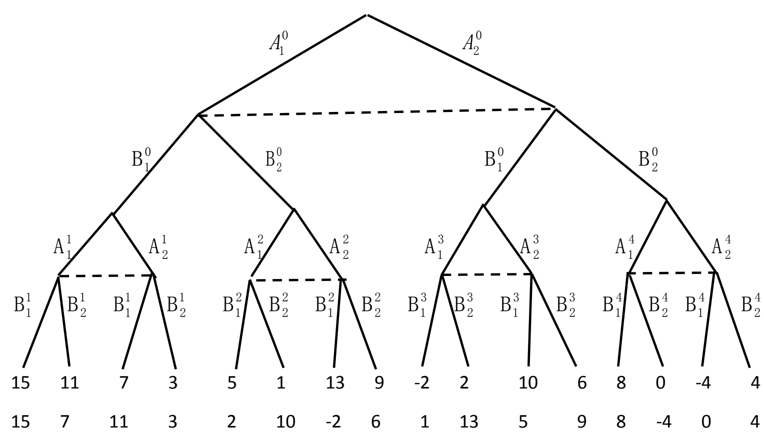

Figure 3: *Example*

Impose the following order on the actions: $A_1^n \prec A_2^n,\ B_1^n \prec B_2^n,\ n = 0,\ 1,\ 2,\ 3,\ 4$. For convenience, we denote the subgame reached when A_1^0 and B_1^0 are played in the first stage Subgame 1. Similarly, we denote from left to right Subgame 2, Subgame 3 and Subgame 4. For player 1, it is easy to see that whenever Subgame 1 is reached on the path of profile, choosing action A_1^1 will yield a strictly higher payoff for player 1 in response to any player 2's choice in that subgame.

When respond to player 2's strategy with B_1^0 played in the first stage, selecting A_1^0 in the first stage to reach subgame 1 then choosing A_1^1 will yield strictly higher payoff for player 1 than any other strategies. Similarly, in response to player 2's strategy with B_2^0 being played in the first stage, player 1 will best respond by playing A_1^0 in the first stage then playing A_2^2 in Subgame 2 and indifferent among the choices in rest of the subgames. It is easy to see that this game exhibits strategic complements.

However, this game does not satisfy quasi-supermodularity conditions and single crossing conditions imposed by Echenique (2004) thus is not an extensive-form game of strategic complementarities. In fact, we showed in Section 2.3 that under differential payoff to outcome assumption, the two-stage games with payoff assignments satisfying the existing conditions have measure zero. However, the measure of two-stage games exhibiting strategic complements with differential payoff at terminal nodes is strictly positive.

2.2 Basic Framework

Consider a two-stage two-player game played in the following way. In the first stage, player 1 and 2 plays game 0. Player 1 chooses between A_1^0 and A_2^0, player 2 chooses in B_1^0 and B_2^0. Depending on the outcome in the first game, a different second-stage game is played. If A_1^0 and B_1^0 is chosen, game 1 is played; if A_1^0 and B_2^0 is chosen, game 2 is played; if A_2^0 and B_1^0 is chosen, game 3 is played and if A_2^0 and B_2^0 is chosen, game 4 is played. And depending on the game played, player 1 chooses between A_1^n and A_2^n

and player 2 chooses between B_1^n and B_2^n, $n = 1, 2, 3, 4$. The set of player 1's strategy is $S = \{(s^0, s^1, s^2, s^3, s^4) : \ s^n \in S_1^n = \{A_1^n, \ A_1^n\}, n = 0, 1, 2, 3, 4\}$ and player 2's strategy set is $T = \{(t^0, t^1, t^2, t^3, t^4) : \ t^n \in S_2^n = \{B_1^n, B_2^n\}, n = 0, 1, 2, 3, 4\}$.

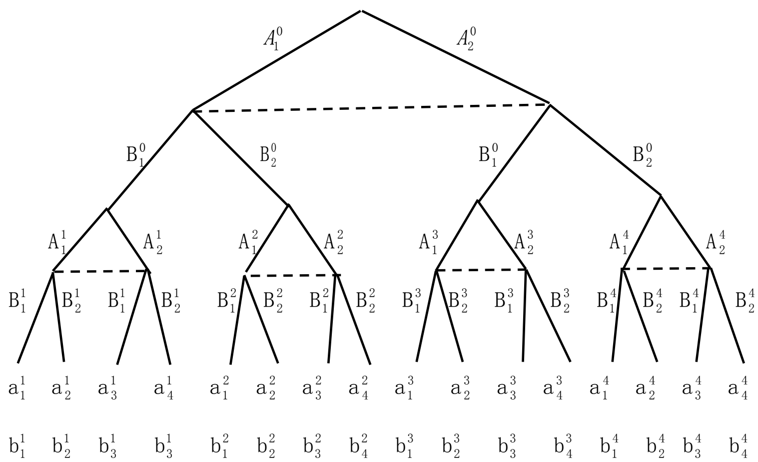

Figure 4: More General Game

Figure 2. is the extensive form Γ of the two-stage game with payoffs assigned at terminal nodes.

Suppose $A_1^n \prec A_2^n$ and $B_1^n \prec B_2^n$ for $n = 0, 1, 2, 3, 4$. Notice here pick any $s, \hat{s} \in S$, $s \prec \hat{s}$ if $\forall \ n \in \{0, 1, 2, 3, 4\}, s^n \preceq \ \hat{s}^n \ and \ \exists \ n' \in \{0, 1, 2, 3, 4\} \ such \ that \ s^{n'} \prec \ \hat{s}^{n'}$. Similarly, $t \prec \hat{s}$ if $\forall \ n \in \{0, 1, 2, 3, 4\}, \ t^n \preceq \hat{t}^n$ and there exist $n' \in \{0, 1, 2, 3, 4\}$ with $t^{n'} \prec \hat{s}^{n'}$.

Definition 1: Player i has strategic complements in Γ if for all subgames, player i's best response sets are increasing in strong set order with respect to player (-i)' strategy in that subgame.

Definition 2: The extensive form game Γ has strategic complements if both players have strategic complements.

Lemma 1. *Consider a game with differential payoffs to outcomes and suppose player 1 exhibits strategic complements.*

For every $\hat{t}, \tilde{t} \in T$*, for every* $\hat{s} \in BR^1(\hat{t})$*, and for every* $\tilde{s} \in BR^1(\tilde{t})$*, if* $\hat{t}^0 = \tilde{t}^0$*, then* $\hat{s}^0 = \tilde{s}^0$*.*

Proof. Notice that the assumption of differential payoffs to outcomes has the following implications for the structure of best responses. For every $t \in T$, and for every $\hat{s}, \tilde{s} \in BR^1(t)$, the subgame reached on the path of play for profile $(\hat{s}, t)$ is the same as the subgame reached on the path of play for profile $(\tilde{s}, t)$. Moreover, the actions played by each player in the subgame reached on the path of play for profile $(\hat{s}, t)$ are the same as the actions played by each player in the subgame reached on the path of play for profile $(\tilde{s}, t)$. Furthermore, every $s \in S$ that has the same actions as $\hat{s}$ on the path of play for profile $(\hat{s}, t)$ is also a member of $BR^1(t)$.

To prove the lemma, fix $\hat{t}, \tilde{t} \in T$, $\hat{s} \in BR^1(\hat{t})$, and $\tilde{s} \in BR^1(\tilde{t})$.

Suppose first that $\hat{t}^0 = \tilde{t}^0 = B_1^0$, and suppose that $\hat{s}^0 = A_1^0$ and $\tilde{s}^0 = A_2^0$. Notice that the structure of the best response of player 1 implies that $\tilde{s}' = (A_2^0, A_2^1, A_2^2, \tilde{s}^3, A_2^4) \in BR^1(\tilde{t})$. Form $\overline{t} = (B_2^0, \tilde{t}^1, \tilde{t}^2, \tilde{t}^3, \tilde{t}^4)$ and consider $\overline{s} \in BR^1(\overline{t})$. Then $\tilde{t} \preceq \overline{t}$, and using strategic complements for player 1, it follows that $\tilde{s}' \vee \overline{s} \in BR^1(\overline{t})$. In particular, subgame 4 is reached with profile $(\tilde{s}' \vee \overline{s}, \overline{t})$, and therefore, $\overline{s}' = (A_2^0, A_1^1, A_1^2, A_1^3, A_2^4) \in BR^1(\overline{t})$. Moreover, $\tilde{t} \preceq \overline{t}$ implies $\overline{s}' = \overline{s}' \wedge \tilde{s}' \in BR^1(\tilde{t})$. Notice that on path of play for profile $(\overline{s}', \tilde{t})$, subgame 3 is reached and the action played by player 1 in subgame 3 is A_1^3.

Consider $\hat{s} \in BR^1(\hat{t})$ and notice that the structure of best response of player 1 implies that $\hat{s}' = (A_1^0, \hat{s}^1, A_1^2, A_1^3, A_1^4) \in BR^1(\hat{t})$. Let $\underline{t} = \hat{t} \wedge \tilde{t}$ and consider $\underline{s} \in BR^1(\underline{t})$. As $\underline{t} \preceq \hat{t}$, strategic complements for player 1 implies that $\underline{s} \wedge \hat{s}' \in BR^1(\underline{t})$. Notice that on path of play for profile $(\underline{s} \wedge \hat{s}', \underline{t})$, subgame 1 is reached, and therefore, the structure of best response for player 1 implies that $\underline{s}' = (A_1^0, \underline{s}^1 \wedge \hat{s}^1, A_2^2, A_2^3, A_2^4) \in BR^1(\underline{t})$. Using $\underline{t} \preceq \tilde{t}$ and strategic complements for player 1 implies that $\underline{s}' \vee \tilde{s}' \in BR^1(\tilde{t})$. Notice that on path of play for profile $(\underline{s}' \vee \tilde{s}', \tilde{t})$, subgame 3 is reached and the action played by player 1 in subgame 3 is A_2^3. As shown above, this is different from the action played by player 1 on path of play for profile $(\overline{s}', \tilde{t})$, contradicting that both $\overline{s}'$ and $\underline{s}' \vee \tilde{s}'$ are best responses of player 1 to $\tilde{t}$. The case where $\hat{s}^0 = A_2^0$ and $\tilde{s}^0 = A_1^0$ is proved similarly.

Now suppose $\hat{t}^0 = \tilde{t}^0 = B_2^0$, and suppose that $\hat{s}^0 = A_1^0$ and $\tilde{s}^0 = A_2^0$. As subgame 2 is reached on path of play for profile $(\hat{s}, \hat{t})$, it follows that $\hat{s}' = (A_1^0, A_1^1, \hat{s}^2, A_1^3, A_1^4) \in BR^1(\hat{t})$. Form $\underline{t} = (B_1^0, \hat{t}^1, \hat{t}^2, \hat{t}^3, \hat{t}^4)$ and consider $\underline{s} \in BR^1(\underline{t})$. Then $\underline{t} \preceq \hat{t}$, and using strategic complements for player 1, it follows that $\hat{s}' \wedge \underline{s} \in BR^1(\underline{t})$. In particular, subgame 1 is reached with profile $(\underline{s} \wedge \hat{s}', \underline{t})$, and therefore, $\underline{s}' = (A_1^0, A_1^1, A_2^2, A_2^3, A_1^4) \in BR^1(\underline{t})$. Moreover, $\underline{t} \preceq \hat{t}$ implies $\underline{s}' = \underline{s}' \vee \hat{s}' \in BR^1(\hat{t})$. Notice that on path of play for profile $(\underline{s}', \hat{t})$, subgame 2 is reached and the action played by player 1 in subgame 3 is A_2^2.

Consider $\tilde{s} \in BR^1(\tilde{t})$ and notice that the structure of best response of player 1 implies that $\tilde{s}' = (A_2^0, A_2^1, A_2^2, A_2^3, \tilde{s}^4) \in BR^1(\tilde{t})$. Let $\overline{t} = \hat{t} \vee \tilde{t}$ and consider $\overline{s} \in BR^1(\overline{t})$. As $\tilde{t} \preceq \overline{t}$, strategic complements for player 1 implies that $\tilde{s}' \vee \overline{s} \in BR^1(\overline{t})$. Notice that on path of play for profile $(\tilde{s}' \vee \overline{s}, \overline{t})$, subgame 4 is reached, and therefore, the structure of best response for player 1 implies that $\overline{s}' = (A_2^0, A_1^1, A_1^2, A_1^3, \overline{s}^4 \vee \tilde{s}^4) \in BR^1(\overline{t})$. Using $\hat{t} \preceq \overline{t}$ and strategic complements for player 1 implies that $\hat{s}' \wedge \overline{s}' \in BR^1(\hat{t})$. Notice that on path of play for profile $(\hat{s}' \wedge \overline{s}', \hat{t})$, subgame 2 is reached and the action played by player 1 in subgame 2 is A_1^2. This is different from the action played by player 1 on path of play for profile $(\underline{s}', \hat{t})$, contradicting that both $\underline{s}'$ and $\hat{s}' \wedge \overline{s}'$ are best responses of player 1 to $\hat{t}$. The case where $\hat{s}^0 = A_2^0$ and $\tilde{s}^0 = A_1^0$ is proved similarly. ∎

Lemma 2. *Consider a game with differential payoffs to outcomes and suppose player 1 exhibits strategic complements.*

(1) If there exists $\hat{t} \in T$ *and* $\hat{s} \in BR^1(\hat{t})$ *such that* $\hat{t}^0 = B_1^0$ *and* $\hat{s}^0 = A_2^0$*, then for every* $t \in T$ *and for every* $s \in BR^1(t)$*,* $s^0 = A_2^0$*.*

(2) If there exists $\hat{t} \in T$ *and* $\hat{s} \in BR^1(\hat{t})$ *such that* $\hat{t}^0 = B_2^0$ *and* $\hat{s}^0 = A_1^0$*, then for every* $t \in T$ *and for every* $s \in BR^1(t)$*,* $s^0 = A_1^0$*.*

Proof. Notice that the assumption of differential payoffs to outcomes implies the following about the structure of best responses: For every $t \in T$, and for every $\hat{s}, \tilde{s} \in BR^1(t)$, $\hat{s}^0 = \tilde{s}^0$. To prove statement (1), fix $\hat{t} \in T$ and $\hat{s} \in BR^1(\hat{t})$ such that $\hat{t}^0 = B_1^0$ and $\hat{s}^0 = A_2^0$. Form $\underline{t} = (B_1^0, B_1^1, B_1^2, B_1^3, B_1^4) \in T$ and let $\underline{s} \in BR^1(\underline{t})$. Then by the previous lemma, $\underline{s}^0 = \hat{s}^0 = A_2^0$. Now fix arbitrarily $t \in T$ and $s \in BR^1(t)$. As $\underline{t} \preceq t$, strategic complements implies that $\underline{s} \vee s \in BR^1(t)$. As $\underline{s}^0 = A_2^0$, it follows that $(\underline{s} \vee s)^0 = A_2^0$. Finally, as noted above, differential payoffs implies that $s^0 = (\underline{s} \vee s)^0 = A_2^0$, as desired. Statement (2) is proved similarly. ∎

Lemma 3. *Consider a game with differential payoffs to outcomes and suppose player 1 exhibits strategic complements.*

(1) If there exists $\hat{t} \in T$ *and* $\hat{s} \in BR^1(\hat{t})$ *such that* $\hat{t}^0 = B_1^0$ *and* $\hat{s}^0 = A_1^0$*, then for every* $t \in T$ *and for every* $s \in BR^1(t)$*, if* $t^0 = B_1^0$ *then* $s^1 = A_1^1$*.*

(2) If there exists $\hat{t} \in T$ *and* $\hat{s} \in BR^1(\hat{t})$ *such that* $\hat{t}^0 = B_1^0$ *and* $\hat{s}^0 = A_2^0$*, then for every* $t \in T$ *and for every* $s \in BR^1(t)$*, if* $t^0 = B_1^0$ *then* $s^3 = A_1^3$*.*

(3) If there exists $\hat{t} \in T$ *and* $\hat{s} \in BR^1(\hat{t})$ *such that* $\hat{t}^0 = B_2^0$ *and* $\hat{s}^0 = A_2^0$*, then for every*

$t \in T$ and for every $s \in BR^1(t)$, if $t^0 = B_2^0$ then $s^4 = A_2^4$.

(4) If there exists $\hat{t} \in T$ and $\hat{s} \in BR^1(\hat{t})$ such that $\hat{t}^0 = B_2^0$ and $\hat{s}^0 = A_1^0$, then for every $t \in T$ and for every $s \in BR^1(t)$, if $t^0 = B_2^0$ then $s^2 = A_2^2$.

Proof. To prove statement (1), fix $\hat{t} \in T$ and $\hat{s} \in BR^1(\hat{t})$ such that $\hat{t}^0 = B_1^0$ and $\hat{s}^0 = A_1^0$. Fix arbitrarily $t \in T$, $s \in BR^1(t)$ such that $t^0 = B_1^0$. By lemma 1, $s^0 = A_1^0$, and therefore, $s' = (A_1^0, s^1, A_1^2, A_1^3, A_1^4) \in BR^1(t)$. Let $\bar{t} = (B_2^0, t^1, t^2, t^3, t^4) \in T$ and $\overline{s} \in BR^1(\bar{t})$. Structure of best responses implies that $\overline{s}' = (\overline{s}^0, A_1^1, \overline{s}^2, A_1^3, \overline{s}^4) \in BR^1(\bar{t})$. Moreover, $t \preceq \bar{t}$ and strategic complements implies that $s' \wedge \overline{s}' \in BR^1(t)$ and consequently, structure of best responses implies that $s^1 = (s' \wedge \overline{s}')^1 = A_1^1$.

To prove statement (2), fix $\hat{t} \in T$ and $\hat{s} \in BR^1(\hat{t})$ such that $\hat{t}^0 = B_1^0$ and $\hat{s}^0 = A_2^0$. Fix arbitrarily $t \in T$, $s \in BR^1(t)$ such that $t^0 = B_1^0$. By lemma 1, $s^0 = A_2^0$, and therefore, $s' = (A_2^0, A_2^1, A_2^2, s^3, A_2^4) \in BR^1(t)$. Let $\bar{t} = (B_2^0, t^1, t^2, t^3, t^4) \in T$ and $\overline{s} \in BR^1(\bar{t})$. By previous lemma, $\overline{s}^0 = A_2^0$, and therefore, $\overline{s}' = (A_2^0, A_1^1, A_1^2, A_1^3, \overline{s}^4) \in BR^1(\bar{t})$. Moreover, $t \preceq \bar{t}$ and strategic complements imply that $(A_2^0, A_1^1, A_1^2, A_1^3, A_2^4 \wedge \overline{s}^4) = s' \wedge \overline{s}' \in BR^1(t)$ and consequently, structure of best responses implies that $s^3 = (s' \wedge \overline{s}')^3 = A_1^3$.

Statements (3) and (4) are proved similarly. ∎

Whenever some subgames are reached on the best response path, Lemma 3 locates the unique action that must be chosen in that subgame to generate strategic complements.

Definition 1. For $m,\ n \in \{1,\ 2,\ 3,\ 4\}$, $i,\ j \in \{1,\ 2\}$, A_i^m ***strictly dominates*** A_j^n if subgame m and n can be reached under the same stage 1 action for player 2, and regardless of which action player 2 plays in subgame n, action A_i^m in subgame m gives player 1 a higher payoff than A_j^n.

Notice that this definition restricts comparison within the same subgame or between subgame 1 and 3 only and between subgame 2 and 4 only. For example, for a comparison between subgame 1 and 3, action A_1^1 strictly dominates action A_1^3 implies that $min\{a_1^1,\ a_2^1\} > \ max\{a_1^3,\ a_3^3\}$.

Theorem 1. *Consider a game with differential payoffs to outcomes. The following are equivalent.*

1. *Player 1 has strategic complements*

2. *Exactly one of the following holds*

   (a) *A_1^1 dominates A_2^1, A_1^3 and A_2^3, and A_2^2 dominates A_1^2, A_1^4 and A_2^4.*

   (b) *A_1^1 dominates A_2^1, A_1^3 and A_2^3, and A_2^4 dominates A_1^4, A_1^2 and A_2^2*

   (c) *A_1^3 dominates A_2^3, A_1^1 and A_2^1, and A_2^4 dominates A_1^4, A_1^2 and A_2^2*

Proof. For this proof, let $\underline{T} = \{t \in T : t^1 = B_1^0\}$ and $\overline{T} = \{t \in T : t^1 = B_2^0\}$. For sufficiency, suppose player 1 has strategic complements.

As case 1, suppose there exists $\hat{t} \in \underline{T}$, there exists $\hat{s} \in BR^1(\hat{t})$ such that $\hat{s}^0 = A_2^0$. Then lemma 3(2) implies that action A_1^3 is strictly dominant for player 1 in subgame 3. In particular, whenever player 2 plays B_1^0 in the first-stage game, player 1 chooses to reach subgame 3 over subgame 1, and to play A_1^3 in subgame 3, regardless of player 2 choice in the second-stage game. Therefore, the minimum payoff to player 1 after playing A_1^3 in subgame 3 must be larger than any given payoff to player 1 in subgame 1. Thus A_1^3 strictly dominates A_1^1 and A_2^1 in subgame 1. Moreover, lemma 2(1) implies that for every $t \in T$ and $s \in BR^1(t)$, if $t^0 = B_2^0$, then $s^0 = A_2^0$, and therefore, lemma 3(3) implies that action A_2^4 is strictly dominant for player 1 in subgame 4. Reasoning as above, A_2^4 strictly dominates A_1^2 and A_2^2 in subgame 2, and therefore, statement 2(c) holds.

As case 2, suppose for every $\hat{t} \in \underline{T}$, for every $\hat{s} \in BR^1(\hat{t})$, $\hat{s} = A_1^0$. Then lemma 3(1) implies that action A_1^1 is strictly dominant for player 1 in subgame 1, and reasoning as above, it follows that A_1^1 strictly dominates A_1^3 and A_2^3 in subgame 3. Now consider $\overline{T}$. As subcase 1, suppose there exists $\tilde{t} \in \overline{T}$, there exists $\tilde{s} \in BR^1(\tilde{t})$ such that $\tilde{s}^0 = A_1^0$. Then lemma 3(4) implies that action A_2^2 is strictly dominant for player 1 in subgame 2, and that A_2^2 strictly dominates A_1^4 and A_2^4 in subgame 2. Therefore, statement 2(a) holds. As subcase 2, suppose for every $\tilde{t} \in \overline{T}$, for every $\tilde{s} \in BR^1(\tilde{t})$, $\tilde{s}^0 = A_2^0$. Then lemma 3(3) implies that action A_2^4 is strictly dominant for player 1 in subgame 4, and that A_2^4 strictly dominates A_1^2 and A_2^2 in subgame 2. Therefore, statement 2(b) holds.

The reasoning above shows that one of the statements 2(a), 2(b), or 2(c) holds. Furthermore, it is easy to check that the inequalities in these three statements imply that no more than one of the statements holds, from which it follows that exactly one of these three statements holds.

For necessity, suppose exactly one of 2(a), 2(b), or 2(c) holds. Suppose statement 2(a) holds. In this case, for every $t \in \underline{T}$, player 2 plays B_1^0, and given that action A_1^1 is strictly dominant for player 1 in subgame 1, and $\min\{a_1^1, a_2^1\} > \max\{a_1^3, a_2^3, a_3^3, a_4^3\}$, it follows that player 1 chooses to reach subgame 1 over subgame 3. Therefore, the structure of best response of player 1 is given by

$$BR^1(t) = \{(A_1^0, A_1^1, s^2, s^3, s^4) \in S : s^n \in \{A_1^n, A_2^n\}, n = 2, 3, 4\}.$$

Notice that this is a sublattice of S. Similarly, for every $t \in \overline{T}$, player 2 plays B_2^0, and given that action A_2^2 is strictly dominant for player 1 in subgame 2 and $\min\{a_3^2, a_4^2\} > \max\{a_1^4, a_2^4, a_3^4, a_4^4\}$, it follows that player 1 chooses to reach subgame 2 over subgame 4, and therefore, the structure of best response of player 1 is given by

$$BR^1(t) = \{(A_1^0, s^1, A_2^2, s^3, s^4) \in S : s^n \in \{A_1^n, A_2^n\}, n = 1, 3, 4\}.$$

Notice that this is a sublattice of S as well. Now consider arbitrary $\hat{t}, \tilde{t} \in T$ such that $\hat{t} \preceq \tilde{t}$. If $\hat{t}^0 = \tilde{t}^0$, then the structure of best responses shows that $BR^1(\hat{t}) = BR^1(\tilde{t})$, and therefore, $BR^1(\hat{t}) \sqsubseteq BR^1(\tilde{t})$. And if $\hat{t}^0 = B_1^0$ and $\tilde{t}^0 = B_2^0$, then from the structure of best responses, it is easy to check that $BR^1(\hat{t}) \sqsubseteq BR^1(\tilde{t})$. Thus, player 1 exhibits strategic complements.

The cases where statement 2(b) or 2(c) holds are proved similarly. ∎

In particular, statement 2(a) is equivalent to the corresponding payoffs $\min\{a_1^1(> a_3^1), a_2^1(> a_4^1)\} > \max\{a_1^3, a_2^3, a_3^3, a_4^3\}$, $\min\{a_3^2(> a_1^2), a_4^2(> a_2^2)\} > \max\{a_1^4, a_2^4, a_3^4, a_4^4\}$. Statement 2(b) is equivalent to $\min\{a_1^1(> a_3^1), a_2^1(> a_4^1)\} > \max\{a_1^3, a_2^3, a_3^3, a_4^3\}$, and $\min\{a_3^4(> a_1^4), a_4^4(> a_2^4)\} > \max\{a_1^2, a_2^2, a_3^2, a_4^2\}$. Statement 2(c) is equivalent to $\min\{a_1^3(> a_3^3), a_2^3(> a_4^3)\} > \max\{a_1^1, a_2^1, a_3^1, a_4^1\}$, and $\min\{a_3^4(> a_1^4), a_4^4(> a_2^4)\} > \max\{a_1^2, a_2^2, a_3^2, a_4^2\}$.

From now on, we drop the assumption of distinct payoffs at terminal nodes. Lemma 4 indicated which subgames will be reached under strategic complementarity assumptions. Lemma 5, 6, 7 and 8 showed how strategic complementarities restrict on-the-path subgames' structures. We will show that after removing the payoff restrictions, strategic complementarities allows richer subgame structures. However, some common structure remains.

A few notations is introduced here to clarity the concept of strategic complementarities in subgame n. Let S^n be the set of all strategies $s \in S$ that allows subgame n to reached on the path of (s,t) for some $t \in T$. Similarly, T^n is defined as the set of all strategies $t \in T$ that allows subgame n to be reached on the path of (s,t) for some $s \in S$. In particular, subgame 1 and subgame 2 can be reached when $s^0 = A_1^0$, thus $S^1 = S^2 = \{s \in S| \ s^0 = A_1^0\}$, also denote $\underline{S} = \{s \in S| \ s^0 = A_1^0\}$. Similarly, it is easy to see that $S^3 = S^4 = \{s \in S| \ s^0 = A_2^0\}$, $T^1 = T^3 = \{t \in T| \ t^0 = B_1^0\}$ and $T^2 = T^4 = \{t \in T| \ t^0 = B_2^0\}$. Denote $\{s \in S| \ s^0 = A_2^0\} = \overline{S}$, $\{t \in T| \ t^0 = B_1^0\} = \underline{T}$ and $\{t \in T| \ t^0 = B_2^0\} = \overline{T}$.

Definition 1b. For $m, \ n \in \{1, \ 2, \ 3, \ 4\}$, $i, \ j \in \{1, \ 2\}$, A_i^m ***weakly dominates*** A_j^n if subgame m and n can be reached under the same stage 1 action for player 2, and action A_i^m in subgame m gives player 1 the same payoff or a higher payoff than A_j^n regardless of what player 2 plays in subgame m and n.

Definition 2a. For $m, \ n \in \{1, \ 2, \ 3, \ 4\}$, $i, \ j, \ s, \ t \in \{1, \ 2\}$, A_i^m ***at*** B_s^m ***strictly dominates*** A_j^n **at** B_t^n if subgame m and n can be reached under the same stage 1 action for player 2, and action A_i^m in subgame m gives player 1 a higher payoff than A_j^n when player 2 plays B_2^m in subgame m and B_t^n in subgame n.

Definition 2b. For $m,\ n \in \{1,\ 2,\ 3,\ 4\}$, $i,\ j,\ s,\ t \in \{1,\ 2\}$, A_i^m ***at*** B_s^m ***weakly dominates*** A_j^n **at** B_t^n if subgame m and n can be reached under the same stage 1 action for player 2, and action A_i^m in subgame m gives player 1 the same payoff or a higher payoff than A_j^n when player 2 plays B_2^m in subgame m and B_t^n in subgame n.

The definition of strategic complements applied to the two-stage game has the following general implications.

Lemma 4. *If player 1 exhibits strategic complements, then for arbitrary $\hat{t},\ \tilde{t} \in \overline{T}$, $\hat{t}^2 \preceq \tilde{t}^2$ and $\hat{t}^4 \preceq \tilde{t}^4$, $BR^1(\hat{t}) \sqsubseteq BR^1(\tilde{t})$. It has the following implications:*

(i). If subgame 4 can be reached on the best response paths to $\hat{t}$, then subgame 4 can be reached on the best response paths to $\tilde{t}$

(ii). If only subgame 4 can be reached on the best response paths to $\hat{t}$, then only subgame 4 can be reached on the best response paths to $\tilde{t}$

(iii). If subgame 2 can be reached on the best response paths to $\tilde{t}$, then subgame 2 can be reached on the best response paths to $\hat{t}$ and subgame 1 can be reached on the best response paths to all $t \in \underline{T}$.

(iv). If only subgame 2 can be reached on the best response path to $\tilde{t}$, then only subgame 2 can be reached on the best response path to $\hat{t}$ and only subgame 1 can be reached on the best response paths to all $t \in \underline{T}$.

Proof. Fix such $\hat{t},\ \tilde{t} \in \overline{T}$. Form $t' = (B_2^0, \hat{t}^1 \vee \tilde{t}^1, \tilde{t}^2, \hat{t}^3 \vee \tilde{t}^3, \tilde{t}^4)$. Since $\hat{t} \preceq t'$, strategic complements implies that $BR^1(\hat{t}) \sqsubseteq BR^1(t')$. Since $\tilde{t}^0 = t'^0 = B_2^0$, only subgame 2 and 4 can be reached with $\tilde{t}$ and t'. Since $t'^2 = \tilde{t}^2$ and $t'^4 = \tilde{t}^4$ in subgame 2 and 4, $BR^1(\tilde{t}) = BR^1(t')$. Thus $BR^1(\hat{t}) \sqsubseteq BR^1(\tilde{t})$.

(i). Fix $\hat{t}$ and $\tilde{t}$. Since there exists $\hat{s} \in BR^1(\hat{t})$ such that $\hat{s}^0 = A_2^0$, pick arbitrary

$\tilde{s} \in BR^1(\tilde{t})$. $BR^1(\hat{t}) \sqsubseteq BR^1(\tilde{t})$ implies that $\tilde{s} \vee \hat{s} \in BR^1(\tilde{t})$ and moreover, $(\tilde{s} \vee \hat{s})^0 = A_2^0$. Thus $BR^1(\tilde{t}) \cap \overline{S} \neq \emptyset$.

(ii). Fix $\hat{t}$ and $\tilde{t}$. Suppose there exists $\tilde{s} \in BR^1(\tilde{t})$ such that $\tilde{s}^0 = A_1^0$. Pick arbitrary $\hat{s} \in BR^1(\hat{t})$. $BR^1(\hat{t}) \sqsubseteq BR^1(\tilde{t})$ implies that $\tilde{s} \wedge \hat{s} \in BR^1(\hat{t})$, in particular, $(\tilde{s} \wedge \hat{s})^0 = A_1^0$. Contradiction. Thus $BR^1(\tilde{t}) \subseteq \overline{S}$.

(iii). Fix $\hat{t}$ and $\tilde{t}$. Pick arbitrary $\tilde{s} \in BR^1(\tilde{t})$, $\tilde{s}^0 = A_1^0$ and $\hat{s} \in BR^1(\hat{t})$. $BR^1(\hat{t}) \sqsubseteq BR^1(\tilde{t})$ implies that $\tilde{s} \wedge \hat{s} \in BR^1(\hat{t})$, moreover, $(\tilde{s} \wedge \hat{s})^0 = A_1^0$. Thus $BR^1(\hat{t}) \cap \underline{S} \neq \emptyset$. Pick arbitrary $t \in \underline{T}$. Form $t' = (B_1^0,\ t^1 \vee\ \tilde{t}^1,\ \tilde{t}^2,\ t^3 \vee\ \tilde{t}^3,\ \tilde{t}^4)$ and $t'' = (B_1^0,\ t^1 \vee\ \tilde{t}^1,\ t^2,\ t^3 \vee\ \tilde{t}^3,\ t^4)$. Since $BR^1(t') = BR^1(t'')$ and $t \prec t''$, $BR^1(t) \sqsubseteq BR^1(t')$. Form $\tilde{t}' = (B_2^0,\ t^1 \vee\ \tilde{t}^1,\ \tilde{t}^2,\ t^3 \vee\ \tilde{t}^3,\ \tilde{t}^4)$. Since $t' \prec \tilde{t}'$ and $BR^1(\tilde{t}) = BR^1(\tilde{t}')$, $BR^1(t') \sqsubseteq BR^1(\tilde{t})$. Thus $BR^1(t) \sqsubseteq BR^1(\tilde{t})$. Pick $s \in BR^1(t)$, $BR^1(t) \sqsubseteq BR^1(\tilde{t})$ implies that $s \wedge \tilde{s} \in BR^1(t)$, moreover, $(s \wedge \tilde{s})^0 = A_1^0$. Thus $BR^1(t) \cap \underline{S} \neq \emptyset$.

(iv). Fix $\hat{t}$ and $\tilde{t}$. Suppose there exists $\hat{s} \in BR^1(\hat{t})$, $\hat{s}^0 = A_2^0$. Then pick arbitrary $\tilde{s} \in BR^1(\tilde{t})$, $\hat{s} \vee \tilde{s} \in BR^1(\tilde{t})$, moreover, $(\hat{s} \vee \tilde{s})^0 = A_2^0$. Contradiction. Thus $BR^1(\hat{t}) \subseteq \underline{S}$. Pick arbitrary $t \in \underline{T}$, form $t' = (B_1^0,\ t^1,\ t^2 \wedge\ \tilde{t}^2,\ t^3,\ t^4 \wedge \tilde{t}^4)$. Thus $BR^1(t) = BR^1(t')$. Form $\tilde{t}' = (B_2^0,\ t^1,\ \tilde{t}^2 \wedge\ t^2,\ t^3,\ \tilde{t}^4 \wedge\ t^4)$ and $\tilde{t}'' = (B_2^0,\ t^1 \wedge\ \tilde{t}^1,\ t^2 \wedge\ \tilde{t}^2,\ t^3 \wedge\ \tilde{t}^3,\ t^4 \wedge\ \tilde{t}^4)$. Since $\tilde{t}'' \prec \tilde{t}$ and $BR^1(\tilde{t}') = BR^1(\tilde{t}'')$, $BR^1(\tilde{t}') \sqsubseteq BR^1(\tilde{t})$. As $t' \prec \tilde{t}'$, strategic complementarities imply that $BR^1(t') \sqsubseteq BR^1(\tilde{t}')$ and $BR^1(t') \sqsubseteq BR^1(\tilde{t})$. Since $BR^1(t) = BR^1(t')$, $BR^1(t) \sqsubseteq BR^1(\tilde{t})$. Suppose there exists $s \in BR^1(t)$, $s^0 = A_2^0$. Pick arbitrary $\tilde{s} \in BR^1(\tilde{t})$. $BR^1(t) \sqsubseteq BR^1(\tilde{t})$ implies $s \vee\ \tilde{s} \in BR^1(\tilde{t})$, moreover, $(s \vee \tilde{s})^0 = A_2^0$. Contradiction. Thus $BR^1(t) \subseteq \underline{S}$. ∎

Corollary 1. *If player 1 exhibits strategic complements, then for arbitrary* $\hat{t},\ \tilde{t} \in \underline{T}$, $\hat{t}^1 \preceq \tilde{t}^1$ *and* $\hat{t}^3 \preceq \tilde{t}^3$, $BR^1(\hat{t}) \sqsubseteq BR^1(\tilde{t})$. *It has the following implications:*

(i). If subgame 1 can be reached on the best response paths to $\tilde{t}$*, then subgame 1 can be*

reached on the best response paths to $\hat{t}$

(ii). If only subgame 1 can be reached on the best response paths to $\tilde{t}$*, then only subgame 1 can be reached on the best response paths to* $\hat{t}$

(iii). If subgame 3 can be reached on the best response paths to $\hat{t}$*, then subgame 3 can be reached on the best response paths to* $\tilde{t}$ *and subgame 3 can be reached on the best response paths to all* $t \in \overline{T}$

(iv). If only subgame 3 can be reached on the best response paths to $\hat{t}$*, then only subgame 3 can be reached on the best response paths to* $\tilde{t}$ *and only subgame 4 can be reached on the best response paths to all* $t \in \overline{T}$*.*

Proof. Proved similarly to Lemma 4.

Lemma 5. *Suppose player 1 exhibits strategic complements.*

(1). If subgame 2 can be reached on the best response paths to some $t \in \overline{T}$*, then* A_2^2 *weakly dominates* A_1^2*.*

(2). If subgame 4 can be reached on the best response paths to some $t \in \overline{T}$*, then* A_2^4 *weakly dominates* A_1^4 *at* t^4*.*

(3). If subgame 4 can be reached on the best response paths to some $t \in \overline{T}$*, then then for every* $\tilde{t} \in \overline{T}$ *such that subgame 2 can be reached on the best response paths,* A_1^2 *weakly dominates* A_2^2 *at* $\tilde{t}^2$*.*

(4). If A_1^4 *weakly dominates* A_2^4 *at* B_1^4 *and there exists* $\hat{t}, \tilde{t} \in \overline{T}$ *such that* $\hat{t}^4 = B_1^4$*,* $\tilde{t}^4 = B_2^4$*,* $\tilde{t}^2 \succeq \hat{t}^2$*, subgame 2 can be reached on the best response paths to* $\tilde{t}$ *and subgame 4 can be reached on the best response path to* $\hat{t}$*, then* A_1^4 *weakly dominates* A_2^4*.*

Proof. (1). Fix $\hat{t} \in \overline{T}$ and $\hat{s} \in BR^1(\hat{t})$ such that $\hat{s}^0 = A_1^0$. Form $\tilde{t} = (B_2^0, \hat{t}^1, B_1^2, \hat{t}^3, \hat{t}^4)$. Since $\tilde{t}^2 \preceq \hat{t}^2$ and $\tilde{t}^4 = \hat{t}^4$, Lemma 4 implies that there exists $\tilde{s} \in BR^1(\tilde{t})$ such that $\tilde{s}^0 = A_1^0$.

Since only subgame 2 is reached on the path of $(\tilde{s}, \tilde{t})$, $\tilde{s}' = (A_1^0, A_2^1, \tilde{s}^2, A_2^3, A_2^4) \in BR^1(\tilde{t})$. Pick arbitrary $\hat{t}' \in \underline{T}$, Lemma 4 implies that there exists $\hat{s}' \in BR^1(\hat{t}')$ such that $\hat{s}'^0 = A_1^0$. Sine subgame 1 is reached on the path of $(\hat{s}', \hat{t}')$, $\hat{s}'' = (A_1^0, \hat{s}'^1, A_2^2, A_2^3, A_2^4) \in BR^1(\hat{t}')$. Lemma 4 implies that $BR^1(\hat{t}') \sqsubseteq BR^1(\tilde{t})$ thus $\hat{s}'' \vee \hat{s}'' = (A_1^0, A_2^1, A_2^2, A_2^3, A_2^4) \in BR^1(\tilde{t})$. Since subgame 2 is reached on the path of $(\hat{s}'' \vee \hat{s}'', \hat{t}')$, $(\hat{s}' \vee \underline{s}')^2 \in BR^{2,1}(\hat{t}^2)$ that is, $A_2^2 \in BR^{2,1}(B_1^2)$. Since player 1 has strategic complementarities in subgame 2, $A_2^2 \in BR^{2,1}(B_2^2)$. Thus A_2^2 is a weakly dominant strategy in subgame 2.

(2). Fix $\hat{t} \in \overline{T}$ such that there exists $\hat{s} \in BR^1(\hat{t})$ and $\hat{s}^0 = A_2^0$. Pick arbitrary $\tilde{t} \in \underline{T}$, Lemma 4(1) implies that $BR^1(\tilde{t}) \sqsubseteq BR^1(\hat{t})$. Pick arbitrary $\tilde{s} \in BR^1(\tilde{t})$. Since only subgame 1 or 3 can be reached on the path of $(\tilde{s}, \tilde{t})$, $\tilde{s}' = (\tilde{s}^0, \tilde{s}^1, A_2^2, \tilde{s}^3, A_2^4) \in BR^1(\tilde{t})$. Thus $\tilde{s}' \vee \hat{s} = (A_2^0, \tilde{s}'^1 \vee \hat{s}^1, A_2^2, \tilde{s}'^3 \vee \hat{s}^3, A_2^4) \in BR^1(\hat{t})$. Since subgame 4 is reached on the path of $(\tilde{s}' \vee \hat{s}, \hat{t})$, $A_2^4 \in BR^{1,4}(\hat{t}^4)$.

(3). Fix $\hat{t} \in \overline{T}$ such that there exists $\hat{s} \in BR^1(\hat{t})$ and $\hat{s}^0 = A_2^0$ and fix $\tilde{t} \in \overline{T}$ such that there exists $\tilde{s} \in BR^1(\tilde{t})$ and $\tilde{s}^0 = A_1^0$. Form $\tilde{t}' = \tilde{t} \vee \hat{t}$, Lemma 5(1) implies that there exists $\tilde{s}' \in BR^1(\tilde{t}')$ such that $\tilde{s}'^0 = A_2^0$. Since subgame 4 is reached on the path of $(\tilde{s}', \tilde{t}')$, $\hat{s}' = (A_2^0, A_1^1, A_1^2, A_1^3, \tilde{s}'^4) \in BR^1(\tilde{t}')$. $\tilde{t} \preceq \tilde{t}'$ and strategic complements in player 1 implies that $\hat{s}' \wedge \tilde{s} = (A_1^0, A_1^1, A_1^2, A_1^3, \tilde{s}^4 \wedge \tilde{s}'^4) \in BR^1(\tilde{t})$. Since subgame 2 is reached on the path of profile $(\tilde{s}' \wedge \tilde{s}, \tilde{t})$, $A_1^2 \in BR^{1,2}(\tilde{t}^2)$.

(4). Since $BR^1(\hat{t}) \cap \overline{S} \neq \emptyset$, there exists $\hat{s} \in BR^1(\hat{t})$ such that $\hat{s}^0 = A_2^0$ and moreover, subgame 4 is reached on the path of $(\hat{s}, \hat{t})$. Since $\hat{t}^4 = B_1^4$ and $A_1^4 \in BR^{1,4}(B_1^4)$, $\hat{s}' = (A_2^0, \hat{s}^1, \hat{s}^2, \hat{s}^3, A_1^4) \in BR^1(\hat{t})$. Since $BR^1(\tilde{t}) \cap \underline{S} \neq \emptyset$, there exists $\tilde{s} \in BR^1(\tilde{t})$ such that $\tilde{s}^0 = A_1^0$. As only subgame 2 is reached on the path of $(\tilde{s}, \tilde{t})$, $\tilde{s}' = (A_1^0, A_2^1, \tilde{s}^2, A_2^3, A_1^4) \in BR^1(\tilde{t})$. Since $\hat{t}^4 \prec \tilde{t}^4$ and $\hat{t}^2 \preceq \tilde{t}^2$, Lemma 4 implies that $BR^1(\hat{t}) \sqsubseteq BR^1(\tilde{t})$. Thus

$\tilde{s}' \vee \hat{s}' = (A_2^0, A_2^1, \tilde{s}^2 \vee \hat{s}^2, A_2^3, A_1^4) \in BR^1(\tilde{t})$. Since subgame 4 is reached on the path of profile $(\tilde{s}' \vee \hat{s}', \tilde{t})$ and $\tilde{t}^4 = B_2^4$, $A_1^4 \in BR^{1,4}(B_2^4)$.∎

Corollary 2. *Suppose player 1 exhibits strategic complements.*

(1). If subgame 3 can be reached on the best response path to some $\hat{t} \in \underline{T}$, then A_1^3 weakly dominates A_2^3

(2). If subgame 1 can be reached on the best response paths to some $\hat{t} \in \underline{T}$, then A_1^1 weakly dominates A_2^1 at $\hat{t}^1$

(3). If subgame 1 can be reached on the best response paths to some $\hat{t} \in \underline{T}$, then for all $\tilde{t} \in \underline{T}$ such that subgame 3 can be reached on the best response paths, A_2^3 weakly dominates A_1^3 at $\tilde{t}^3$

(4). If A_2^1 weakly dominates A_1^1 at B_2^1 and there exists $\hat{t}, \tilde{t} \in \underline{T}$ such that $\hat{t}^1 = B_2^1$, $\tilde{t}^1 = B_1^1$, $\tilde{t}^3 \preceq \hat{t}^3$, subgame 1 can be reached on the best response paths to $\hat{t}$ and subgame 3 can be reached on the best response path to $\tilde{t}$, then A_2^1 weakly dominates A_1^1.

Proof. Proved similarly as Lemma 5

Lemma 6. *Suppose player 1 exhibits strategic complements, if subgame 3 can be reached on the best response paths of some $\hat{t} \in \underline{T}$ and subgame 2 can be reached on the best response paths of some $\tilde{t} \in \overline{T}$, then*

i. $BR^1(\tilde{t}) = BR^1(\hat{t}) = S$.

ii. for $t' \in \{t \in \overline{T} | \ t^2 = B_1^2, \ t^4 = B_1^4\} \cup \{t \in \underline{T} | \ t^1 = B_2^1, \ t^3 = B_2^3\}$, $BR^1(t') = S$.

Proof. i. Fix $\hat{t} \in \underline{T}$ and $\tilde{t} \in \overline{T}$ in the assumption. Then there exists $\tilde{s} \in BR^1(\tilde{t})$ such that $\tilde{s}^0 = A_1^0$, in particular, $\tilde{s}' = (A_1^0, A_2^1, \tilde{s}^2, A_1^3, A_1^4) \in BR^1(\tilde{t})$. And there exists $\hat{s} \in BR^1(\hat{t})$ such that $\hat{s}^0 = A_2^0$, in particular, $\hat{s}' = (A_2^0, A_2^1, A_2^2, \hat{s}^3, A_1^4) \in BR^1(\hat{t})$.

Form $\tilde{t}' = (B_2^0, \hat{t}^1, \tilde{t}^2, \hat{t}^3, \tilde{t}^4)$ and $\hat{t}' = (B_1^0, \hat{t}^1, \tilde{t}^2, \hat{t}^3, \tilde{t}^4)$. Thus $BR^1(\tilde{t}') = BR^1(\tilde{t})$ and $BR^1(\hat{t}') = BR^1(\hat{t})$. $\tilde{t}' \succ \hat{t}'$ and strategic complements in player 1 implies that $\tilde{s}' \vee \hat{s}' = (A_2^0, A_2^1, A_2^2, \hat{s}^3, A_1^4) \in BR^1(\tilde{t}') = BR^1(\tilde{t})$ and $\tilde{s}' \wedge \hat{s}' = (A_1^0, A_2^1, \tilde{s}^2, A_1^3, A_1^4) \in BR^1(\hat{t}') = BR^1(\hat{t})$. Thus $A_1^4 \in BR^{4,1}(\tilde{t}^4)$ and $A_2^1 \in BR^{1,1}(\hat{t}^1)$. Lemma 5(2) implies $A_2^4 \in BR^{4,1}(\tilde{t}^4)$, thus $\overline{S} \subseteq BR^1(\tilde{t})$. Similarly, as $A_1^1 \in BR^{1,1}(\hat{t}^1)$, $\underline{S} \subseteq BR^1(\hat{t})$.

Since $\underline{S} \cap BR^1(\tilde{t}) \neq \emptyset$ and $\overline{S} \cap BR^1(\tilde{t}) \neq \emptyset$, Lemma 5 implies $A_1^2,\ A_2^2 \in BR^{2,1}(\tilde{t})$. Thus $\underline{S} \subseteq BR^1(\tilde{t})$. So $S = BR^1(\tilde{t})$. Similarly, $S = BR^1(\hat{t})$.

ii. It is implied by Lemma 4 and Corollary 1. ∎

Lemma 7. *Suppose player 1 exhibits strategic complements. For all $t \in T$ and $s \in BR^1(t)$ such that subgame n is reached on the path of (s, t),*

(1). If A_2^n weakly dominates A_1^n at B_1^n, then A_2^n weakly dominates A_1^n.

(2). If A_1^n weakly dominates A_2^n at B_2^n, then A_1^n weakly dominates A_2^n.

Proof. (1). Since $s \in BR^1(t)$ and subgame n is reached on the path of (s, t), $A_2^n \in BR^{n,1}(B_1^n)$. Pick arbitrary $s^n \in BR^{n,1}(B_2^n)$. As player 1 has strategic complements in subgame n and $B_1^n \prec B_2^n$, $A_2^n \in BR^{n,1}(B_2^n)$.

(2). Proved Similarly as (1). ∎

Lemma 8 is the additional best response structure inherited from two-stage game structure only.

Lemma 8. *Suppose player 1 has strategic complements, if subgame 2 can be reached on the best response path of some $t \in \{t \in \overline{T} |\ t^2 = B_2^2,\ t^4 = B_2^4\}$, and subgame 4 can be reached on the best response paths of some $\hat{t} \in \{t \in \overline{T} |\ t^2 = B_1^2,\ t^4 = B_2^4\}$ and $\tilde{t} \in \{t \in \overline{T} |\ t^2 = B_2^2,\ t^4 = B_1^4\}$, then* *i. player 1 is indifferent between A_1^2, A_2^2 and A_2^4*

ii. A_1^2, A_2^2 and A_2^4 weakly dominate A_1^4.

Proof. Since $\hat{t}^2 \prec t^2$ and $\hat{t}^4 \prec t^4$, Lemma 4 implies subgame 2 can be reached on the best response path of $\hat{t}$ and subgame 4 can be reached on the best response paths of t. Lemma 5 implies that $a_1^2 = a_3^2 = a_4^4 \geq a_2^4$ and $a_2^2 = a_4^2 = a_4^4 \geq a_2^4$. Thus $a_1^2 = a_3^2 = a_2^2 = a_4^2 = a_4^4 \geq a_2^4$. Similarly, for $\tilde{t}$, subgame 2 can be reached on the best response path and $a_2^2 = a_4^2 = a_3^4 \geq a_1^4$. Thus $a_1^2 = a_3^2 = a_2^2 = a_4^2 = a_4^4 = a_3^4 \geq max\{a_2^4, a_1^4\}$. Thus i and ii are proved. ∎

Corollary 3. *Suppose player 1 has strategic complements, if subgame 4 can be reached on the best response path of some $t \in \{t \in \overline{T}|\ t^2 = B_1^2,\ t^4 = B_1^4\}$, and subgame 2 can be reached on the best response paths of some $\hat{t} \in \{t \in \overline{T}|\ t^2 = B_1^2,\ t^4 = B_2^4\}$ and $\tilde{t} \in \{t \in \overline{T}|\ t^2 = B_2^2,\ t^4 = B_1^4\}$, then*

i. player 1 is indifferent between A_1^2, A_2^2 and A_2^4

ii. A_1^2, A_2^2 and A_2^4 weakly dominate A_1^4.

Corollary 4. *If both subgame 1 and 3 can be reached on the best response path for all t in $\underline{T}\backslash\{t \in T|\ t^1 = B_1^1,\ t^3 = B_1^3\}$, then both subgames can be reached on the best response path for all t in $\underline{T}$. Similarly for all $t \in \underline{T}\backslash\{t \in T|\ t^1 = B_2^1,\ t^3 = B_2^3\}$.*

Now let's study exactly under what conditions the 2-stage 2×2 game can exhibit strategic complements.

Case 1: Suppose only subgame 1 can be reached on the best response paths to all $t \in \underline{T}$. Corollary 2 indicates that A_1^1 weakly dominates A_2^1 and strongly dominates A_1^3 and A_2^3.

A.

Between subgame 1 and 3:

$a_4^1 = a_2^1$ *and* $a_3^1 = a_1^1$ *or* $a_1^1 > a_3^1$ *and* $a_2^1 \geq a_4^1$

$a_1^1(\geq a_3^1), a_2^1(\geq a_4^1) > \max\{a_1^3, a_2^3, a_3^3, a_4^3\}$

And between subgame 2 and 4

(a1) subgame 4: $a_4^4 = a_2^4$ *and* $a_3^4 = a_1^4$ *or* $a_4^4 > a_2^4$ *and* $a_3^4 \geq a_1^4$

$a_4^4(\geq a_2^4), a_3^4(\geq a_1^4) > \max\{a_1^2, a_2^2, a_3^2, a_4^2\}$

Or

(a2) subgame 2: $a_4^2 = a_2^2$ *and* $a_3^2 = a_1^2$ *or* $a_4^2 > a_2^2$ *and* $a_3^2 \geq a_1^2$

$a_4^2(\geq a_2^2), a_3^2(\geq a_1^2) > \max\{a_1^4, a_2^4, a_3^4, a_4^4\}$

Or

(a3) subgame 4: $a_4^4 = a_2^4$ *and* $a_3^4 = a_1^4$ *or* $a_4^4 > a_2^4$ *and* $a_3^4 \geq a_1^4$

$a_1^2 = a_3^2 > a_4^4(\geq a_2^4) > a_2^2 = a_4^2 > a_3^4(\geq a_1^4)$ or

$a_4^4(\geq a_2^4) > a_2^2 = a_4^2, a_1^2 = a_3^2 > a_3^4(\geq a_1^4)$ or

$a_4^4(\geq a_2^4) > a_1^2 = a_3^2 > a_3^4(\geq a_1^4) > a_4^2(\geq a_2^2)$ or

$a_1^2 = a_3^2 > a_4^4(\geq a_2^4), a_3^4(\geq a_1^4) > a_4^2(\geq a_2^2)$

Or

(a4) subgame 4: $a_4^4 = a_2^4$ *and* $a_3^4 = a_1^4$ *or* $a_4^4 > a_2^4$ *and* $a_3^4 \geq a_1^4$

$a_4^4(\geq a_2^4) > a_1^2 = a_3^2 = a_3^4(\geq a_1^4) > a_4^2(\geq a_2^2)$ or

$a_1^2 = a_3^2 = a_4^4(\geq a_2^4) > a_2^2 = a_4^2 > a_3^4(\geq a_1^4)$ or

$a_1^2 = a_3^2 = a_4^4(\geq a_2^4) > a_3^4(\geq a_1^4) > a_4^2(\geq a_2^2)$ or

$a_1^2 = a_3^2 > a_4^4(\geq a_2^4) > a_3^4(\geq a_1^4) = a_2^2 = a_4^2$ or

$a_4^4(\geq a_2^4) > a_1^2 = a_3^2 > a_3^4(\geq a_1^4) = a_2^2 = a_4^2$ or

$a_1^2 = a_3^2 > a_2^2 = a_4^2 = a_4^4(\geq a_2^4) > a_3^4(\geq a_1^4)$ or

$a_1^2 = a_3^2 = a_3^4(\geq a_1^4) = a_4^4(\geq a_2^4) > a_4^2(\geq a_2^2)$ *and if* $a_1^4 = a_3^4$ *then* $a_2^4 = a_4^4$ or

$a_4^4(\geq a_2^4) > a_1^2 = a_3^2 = a_2^2 = a_4^2 = a_3^4(\geq a_1^4)$ or

$a_1^2 = a_3^2 = a_4^4(\geq a_2^4) > a_2^2 = a_4^2 = a_3^4(\geq a_1^4)$ or

$a_1^2 = a_2^2 = a_3^2 = a_4^2 = a_4^4(\geq a_2^4) > a_3^4(\geq a_1^4)$ or

$a_1^2 = a_3^2 > a_2^2 = a_4^2 = a_4^4(\geq a_2^4) = a_3^4(\geq a_1^4)$ *and if* $a_1^4 = a_3^4$ *then* $a_2^4 = a_4^4$ or

$a_1^2 = a_3^2 = a_2^2 = a_4^2 = a_4^4(\geq a_2^4) = a_3^4(\geq a_1^4)$ *and if* $a_1^4 = a_3^4$ *then* $a_2^4 = a_4^4$

Figure 5: *Case* 1

i. if only subgame 4 can be reached on the best response paths to all $t \in \overline{T}$, then Lemma 5 indicates that A_2^4 weakly dominates A_1^4 and strongly dominates A_1^2 and A_2^2.

ii. if only subgame 2 can be reached on the best response paths to all $t \in \overline{T}$, then Lemma 5 indicates that A_2^2 weakly dominates A_1^2 and strongly dominates A_1^4 and A_2^4.

iii. if subgame 2 can be reached on the best response paths to some $t \in \overline{T}$ and subgame 4 can be reached on the best response paths to some other $\hat{t} \in \overline{T}$ but never at the same time. Lemma 4 indicates that all $t \in \{t \in \overline{T} | \ t^2 = B_1^2, \ t^4 = B_1^4\}$, only subgame 2 can be reached on the best response path and for all $t \in \{t \in \overline{T} | \ t^2 = B_2^2, \ t^4 = B_2^4\}$, only subgame 4 can be reached on the best response paths. Lemma 5 implies that player 1 is indifferent between A_1^2 and A_2^2 at B_1^2. And A_1^2 and A_2^2 at B_1^2 strongly dominates A_1^4 and A_2^4 at B_1^4. Lemma 5 also implies that A_2^4 weakly dominates A_1^4 at B_2^4 and strongly dominates A_1^2 and A_2^2 at B_2^2.

iv. if both subgame 2 and 4 can be reached on the best response paths to some $t \in \overline{T}$, then Lemma 4 indicates that for all $t \in \{t \in \overline{T} | \ t^2 = B_1^2, \ t^4 = B_1^4\}$, subgame 2 can be reached on the best response path and for all $t \in \{t \in \overline{T} | \ t^2 = B_2^2, \ t^4 = B_2^4\}$, subgame 4 can be reached on the best response paths. Lemma 5 implies that player 1 is indifferent between A_1^2 and A_2^2 at B_1^2. And A_1^2 and A_2^2 at B_1^2 weakly dominates A_1^4 and A_2^4 at B_1^4. Lemma 5 also implies that A_2^4 weakly dominates A_1^4 at B_2^4 and weakly dominates A_1^2 and A_2^2 at B_2^2. If subgame 2 is reached on the path, player 1 is indifferent among A_1^2 and A_2^2. Similarly, if subgame 4 is reached on the best response paths, A_2^4 weakly dominates A_1^4.

Lemma 7 restricts payoffs within subgames to ensure strategic complements in subgames. The possible payoff assignments of Case 1 are included in Table 1.

Case 2. Suppose only subgame 3 can be reached on the best response paths to some $t \in \underline{T}$ and only subgame 4 can be reached on the best response paths to all $t \in \overline{T}$. Lemma 5 indicates A_2^4 weakly dominates A_1^4 and strongly dominates A_1^2 and A_2^2.

B.

Between subgame 2 and 4

$subgame\ \ 4{:}\ \ a_4^4 = a_2^4\ \ and\ \ a_3^4 = a_1^4\ \ or\ \ a_4^4 > a_2^4\ \ and\ \ a_3^4 \geq a_1^4$

$a_4^4(\geq a_2^4), a_3^4(\geq a_1^4) > \max\{a_1^2, a_2^2, a_3^2, a_4^2\}$

And between subgame 1 and 3

(b1) $subgame\ \ 3{:}\ \ a_1^3 = a_3^3\ \ and\ \ a_2^3 = a_4^3\ \ or\ \ a_1^3 > a_3^3\ \ and\ \ a_2^3 \geq a_4^3$

$a_1^3(\geq a_3^3), a_2^3(\geq a_4^3) > \max\{a_1^1, a_2^1, a_3^1, a_4^1\}$

Or

(b2) $subgame\ \ 1{:}\ \ a_3^1 = a_1^1\ \ and\ \ a_4^1 = a_2^1\ \ or\ \ a_1^1 > a_3^1\ \ and\ \ a_2^1 \geq a_4^1$

$a_1^1(\geq a_3^1) > a_4^3 = a_2^3 > a_2^1(\geq a_4^1) > a_1^3(\geq a_3^3)$ or

$a_4^3 = a_2^3 > a_2^1(\geq a_4^1), a_1^1(\geq a_3^2) > a_1^3(\geq a_3^3)$ or

$a_4^3 = a_2^3 > a_1^1(\geq a_3^1) > a_3^3 = a_1^3 > a_2^1(\geq a_4^1)$ or

$a_1^1(\geq a_3^1) > a_4^3 = a_2^3, a_3^3 = a_1^3 > a_2^1(\geq a_4^1)$

Or

(b3) $subgame\ \ 1{:}\ \ a_3^1 = a_1^1\ \ and\ \ a_4^1 = a_2^1\ \ or\ \ a_1^1 > a_3^1\ \ and\ \ a_2^1 \geq a_4^1$

$a_4^3 = a_2^3 > a_1^1(\geq a_3^1) = a_3^3 = a_1^3 > a_2^1(\geq a_4^1)$ or

$a_1^1(\geq a_3^1) = a_4^3 = a_2^3 > a_2^1(\geq a_4^1) > a_1^3(\geq a_3^3)$ or

$a_1^1(\geq a_3^1) = a_4^3 = a_2^3 > a_3^3 = a_1^3 > a_2^1(\geq a_4^1)$ or

$a_1^1(\geq a_3^1) > a_4^3 = a_2^3 > a_3^3 = a_1^3 = a_2^1(\geq a_4^1)$ or

$a_4^3 = a_2^3 > a_1^1(\geq a_3^1) > a_3^3 = a_1^3 = a_2^2(\geq a_4^2)$ or

$a_1^1(\geq a_3^1) = a_3^3 = a_1^3 = a_4^3 = a_2^3 > a_2^1(\geq a_4^1)$ or

$a_4^3 = a_2^3 > a_1^1(\geq a_3^1) = a_2^1(\geq a_4^1) = a_3^3 = a_1^3\ \ and\ \ if\ \ a_2^1 = a_4^1\ \ then\ \ a_1^1 = a_3^1$ or

$a_1^1(\geq a_3^1) = a_4^3 = a_2^3 > a_2^1(\geq a_4^1) = a_3^3 = a_1^3$

Table 1: *Case* 2

i. if only subgame 3 can be reached on the best response paths to all $t \in \underline{T}$, then Corollary 2 indicates A_1^3 weakly dominates A_2^3 and strongly dominates A_1^1 and A_2^1.

ii. if subgame 1 can be reached on the best response paths to some $t \in \underline{T}$ and subgame 3 can be reached on the best response paths to some other $\hat{t} \in \overline{T}$, but never at the same time. Corollary 1 indicates that all $t \in \{t \in \underline{T}|\ t^1 = B_1^1,\ t^3 = B_1^3\}$, only subgame 1 can be reached on the best response path and for all $t \in \{t \in \underline{T}|\ t^1 = B_2^1,\ t^3 = B_2^3\}$, only subgame 3 can be reached on the best response paths. Corollary 2 implies that player 1 is indifferent between A_1^3 and A_2^3 at B_2^3. And A_1^3 and A_2^3 at B_2^3 strongly dominates A_1^1 and A_2^1 at B_2^1. Lemma 5 also implies that A_1^1 weakly dominates A_2^1 at B_1^1 and strongly dominates A_1^3 and A_2^3 at B_1^3.

iii. if both subgame 1 and 3 can be reached on the best response paths to some $t \in \underline{T}$, then Corollary 1 indicates that for all $t \in \{t \in \underline{T}|\ t^1 = B_1^1,\ t^3 = B_1^1\}$, subgame 1 can be reached on the best response path and for all $t \in \{t \in \underline{T}|\ t^1 = B_2^1,\ t^3 = B_2^3\}$, only subgame 3 can be reached on the best response paths. Corollary 2 implies that player 1 is indifferent between A_1^3 and A_2^3 at B_2^3. And A_1^3 and A_2^3 at B_2^3 strongly dominates A_1^1 and A_2^1 at B_2^1. Corollary 2 also implies that A_1^1 weakly dominates A_2^1 at B_1^1 and weakly dominates A_1^3 and A_2^3 at B_1^3. Moreover, if subgame 3 is reached on the best response path to some $t \in \underline{T}$, player 1 is indifferent among A_1^3 and A_2^3 at t^3. Similarly, if subgame 1 is reached on the best response paths at some $t \in \underline{T}$, A_1^1 weakly dominates A_2^1 at t^1.

Lemma 7 restricts payoffs within subgames to ensure strategic complements in subgames. The possible payoff assignments of Case 2 are included in Table 2.

Case 3. Suppose subgame 3 can be reached on the best response paths to some $t \in \underline{T}$, for all of those t, subgame 1 can also be reached on the best response paths.

i. if only subgame 4 can be reached on the best response paths for all $t \in \overline{T}$, thus Lemma 5 indicates that A_2^2 weakly dominates A_1^2 and strongly dominates A_1^4 and A_2^4. Corollary 1 indicates that for all $t \in \{t \in \underline{T}|\ t^1 = B_2^1,\ t^3 = B_2^3\}$, both subgame 1 and subgame 3 can be reached on the best response paths and for all $t \in \{t \in \underline{T}|\ t^1 = B_1^1,\ t^3 =$

c.

(c1) *Between subgame 2 and 4:*

$subgame\ 4\colon\ a_4^4 = a_2^4\ \ and\ \ a_3^4 = a_1^4\ \ or\ \ a_4^4 > a_2^4\ \ and\ \ a_3^4 \geq a_1^4$

$a_4^4(\geq a_2^4), a_3^4(\geq a_1^4) > \max\{a_1^2, a_2^2, a_3^2, a_4^2\}$

And between subgame 1 and 3:

$subgame\ 1\colon\ a_3^1 = a_1^1\ \ and\ \ a_4^1 = a_2^1\ \ or\ \ a_1^1 > a_3^1\ \ and\ \ a_2^1 \geq a_4^1$

$a_1^1(\geq a_3^1) > a_2^1(\geq a_4^1) = a_4^3 = a_2^3 > a_1^3(\geq a_3^3)$ or

$a_1^1(\geq a_3^1) = a_2^1(\geq a_4^1) = a_4^3 = a_2^3 > a_1^3(\geq a_3^3)$ or

$a_1^1(\geq a_3^1) > a_2^1(\geq a_4^1) = a_4^3 = a_2^3 = a_1^3 = a_3^3\ \ and\ \ if\ \ a_2^1 = a_4^1\ \ then\ \ a_1^1 = a_3^1$ or

$a_1^1(\geq a_3^1) = a_2^1(\geq a_4^1) = a_1^3 = a_2^3 = a_3^3 = a_4^3\ \ and\ \ if\ \ a_2^1 = a_4^1\ \ then\ \ a_1^1 = a_3^1$

Or

(c2) *Between subgame 1 and 3:*

$a_1^1 = a_3^1 > a_2^1 = a_4^1 = a_4^3 = a_2^3 > a_1^3(\geq a_3^3)$ or

$a_1^1 = a_3^1 = a_2^1 = a_4^1 = a_4^3 = a_2^3 > a_1^3(\geq a_3^3)$ or

$a_1^1 = a_3^1 > a_2^1 = a_4^1 = a_4^3 = a_2^3 = a_1^3 = a_3^3$ or

$a_1^1 = a_2^1 = a_3^1 = a_4^1 = a_1^3 = a_2^3 = a_3^3 = a_4^3$

And between subgame 2 and 4:

$a_4^4 = a_2^4 > a_1^2 = a_3^2 = a_3^4 = a_1^4 > a_4^2(\geq a_2^2)$ or

$a_1^2 = a_3^2 = a_3^4 = a_1^4 = a_4^4 = a_2^4 > a_4^2(\geq a_2^2)$ or

$a_4^4 = a_2^4 > a_1^2 = a_3^2 = a_2^2 = a_4^2 = a_3^4 = a_1^4$ or

$a_1^2 = a_2^2 = a_3^2 = a_4^2 = a_1^4 = a_2^4 = a_3^4 = a_4^4$

Table 2: $Case\ 3$

$B_1^3\}$, subgame 1 can be reached on the best response paths. One immediate result obtained from Corollary 2 is that A_1^1 weakly dominates A_2^1, A_1^3 and A_2^3. The dominance between A_1^1 and A_1^3, A_2^3 can be strict if only subgame 1 can be reached on the best response path. Corollary 2 also implies that player 1 is indifferent among A_1^3 and A_2^3 at B_2^3 and A_1^1 at B_2^1.

ii. if subgame 2 can be reached on the best response paths for some $t \in \overline{T}$, then Lemma 4 indicates that for all $t \in \{t \in \overline{T} |\ t^2 = B_1^2,\ t^4 = B_1^4\}$, subgame 2 can be reached on the best response path and for all $t \in \{t \in \overline{T} |\ t^2 = B_2^2,\ t^4 = B_2^4\}$, subgame 4 can be reached on the best response paths. Lemma 5 implies that player 1 is indifferent between A_1^2 and A_2^2 at B_1^2. And A_1^2 and A_2^2 at B_1^2 weakly dominates A_1^4 and A_2^4 at B_1^4. Lemma 5 also implies that A_2^4 weakly dominates A_1^4 at B_2^4 and weakly dominates A_1^2 and A_2^2 at B_2^2. The dominance can be strict, if only one subgame is reached on the best response paths. In that case, if subgame 2 is reached on the path, player 1 is indifferent among A_1^2 and A_2^2. Similarly, if subgame 4 is reached on the best response paths, A_2^4 weakly dominates A_1^4. Corollary 1 indicates that for all $t \in \{t \in \underline{T} |\ t^1 = B_2^1,\ t^3 = B_2^3\}$, both subgame 1 and subgame 3 can be reached on the best response paths and for all $t \in \{t \in \underline{T} |\ t^1 = B_1^1,\ t^3 = B_1^3\}$, subgame 1 can be reached on the best response paths. One immediate result obtained from Corollary 2 is that A_1^1 weakly dominates A_2^1, A_1^3 and A_2^3. The dominance between A_1^1 and A_1^3, A_2^3 can be strict if only subgame 1 can be reached on the best response path. Corollary 2 also implies that player 1 is indifferent among A_1^3 and A_2^3 at B_2^3 and A_1^1 at B_2^1.

Lemma 7 restricts payoffs within subgames to ensure strategic complements in subgames. The possible payoff assignments of Case 3 are included in Table 3.

We can also show that with the payoff assignments, the two-stage game exhibits strategic complements. For example,

Suppose Case 1 holds. For every $\underline{t} \in \underline{T}$, $BR^1(\underline{t}) = \{(A_1^0, \hat{s}^1, s^2, s^3, s^4)|\ \hat{s}^1 \in BR_1^1(\underline{t}^1),\ s^n \in \{A_1^n, A_2^n\}, n \in \{2,3,4\}\}$ and $BR^{1,1}(t^1)$ is increasing in strong set order with respect to $t^1 \in \{B_1^1, B_2^1\}$. Thus for every $\underline{t}, \underline{t}' \in \underline{T}$, if $\underline{t}^1 = \underline{t}'^1$ then $BR^1(\underline{t}) = BR^1(\underline{t}')$ and if $\underline{t}^1 \prec \underline{t}'^1$ then $BR^1(\underline{t}) \sqsubseteq BR^1(\underline{t}')$.

In i, for all $\bar{t} \in \overline{T}$, $BR^1(\bar{t}) = \{(A_2^0, s^1, s^2, s^3, \hat{s}^4)|\ \hat{s}^4 \in BR^{4,1}(\bar{t}^4),\ s^n \in \{A_1^n, A_2^n\}, n \in \{1,2,3\}\}$ and $BR^{4,1}(t^4)$ is increasing in strong set order with respect to $t^4 \in \{B_1^4, B_2^4\}$. Thus for every $\bar{t}, \bar{t}' \in \overline{T}$, if $\bar{t}^4 = \bar{t}'^4$ then $BR^1(\bar{t}) = BR^1(\bar{t}')$ and if $\bar{t}^4 \prec \bar{t}'^4$ then $BR^1(\bar{t}) \sqsubseteq BR^1(\bar{t}')$. For arbitrary $\underline{t} \in \underline{T}$ and $\bar{t} \in \overline{T}$, $BR^1(\underline{t}) \subset \underline{S}$ and $BR^1(\bar{t}) \subset \overline{S}$ and since $A_1^1 \in BR^{1,1}(t^1)$ for $t^1 \in \{B_1^1, B_2^1\}$ and $A_2^4 \in BR^{4,1}(t^4)$ for $t^4 \in \{B_1^4, B_2^4\}$, $BR^1(\underline{t}) \sqsubseteq BR^1(\bar{t})$. Thus player 1 exhibits strategic complements.

All the other cases can be analysed in the same way.

2.3 Compare with Previous Work

In Echenique (2004), an extensive form game with strategic complementarities is defined as a game with payoffs that satisfies quasi-supermodularity conditions and single crossing conditions in all the subgames.

The payoff function $U_i(s,t) : S \times T \to \mathbf{R}$ is ***quasisupermodular*** if for all $t \in T$ and for any $s, s' \in S$, $u(s \wedge s', t) < u(s,t)$ implies $u(s',t) < u(s \vee s', t)$ and $u(s \wedge s', t) \le u(s,t)$ implies $u(s',t) \le u(s \vee s', t)$. The definition is equivalent to $u(s',t) \ge u(s \vee s', t)$ implies $u(s \wedge s', t) \ge u(s,t)$ and $u(s',t) > u(s \vee s', t)$ implies $u(s \wedge s', t) > u(s,t)$.

Claim 1. *Suppose $U_1(s,t)$ is quasisupermodular, then A_1^1, A_1^2, A_2^3 and A_2^4 are weakly* *dominant strategy in subgame 1, subgame 2, subgame 3 and subgame 4 respectively.*

Proof. Pick arbitrary $\overline{s} \in \overline{S}$ and $\underline{s} \in \underline{S}$ such that $\overline{s}^3 = \underline{s}^3$, $\overline{s}^1 = A_1^1$ and $\underline{s}^1 \neq \overline{s}^1$ and pick arbitrary $\underline{t}, \underline{t}' \in \underline{T}$ such that $\underline{t}^3 = \underline{t}'^3$, $\underline{t}^1 = B_1^1$ and $\underline{t}'^1 \neq \underline{t}^1$.

$U_1(\overline{s}, \underline{t}) = U_1(\underline{s} \vee \overline{s}, \underline{t})$ and quasisupermodularity implies $U_1(\underline{s} \wedge \overline{s}, \underline{t}) \geq U_1(\underline{s}, \underline{t})$, that is, $a_1^1 \geq a_3^1$. Similarly, $U_1(\overline{s}, \underline{t}') = U_1(\underline{s} \vee \overline{s}, \underline{t}')$ and quasisupermodularity implies $U_1(\underline{s} \wedge \overline{s}, \underline{t}') \geq U_1(\underline{s}, \underline{t}')$, that is, $a_2^1 \geq a_4^1$. Whenever subgame 1 is reached on the path if arbitrary profile (s, t), A_1^1 offers higher payoff to player 1 in subgame 1. Thus A_1^1 is a weakly dominant strategy in subgame 1.

Pick arbitrary $\overline{s} \in \overline{S}$ and $\underline{s} \in \underline{S}$ such that $\overline{s}^4 = \underline{s}^4$, $\overline{s}^2 = A_1^2$ and $\underline{s}^2 \neq \overline{s}^2$ and pick arbitrary $\overline{t}, \overline{t}' \in \overline{T}$ such that $\overline{t}^4 = \overline{t}'^4$, $\overline{t}^2 = B_1^2$ and $\overline{t}'^2 \neq \overline{t}^2$.

$U_1(\overline{s}, \overline{t}) = U_1(\underline{s} \vee \overline{s}, \overline{t})$ and quasisupermodularity implies $U_1(\underline{s} \wedge \overline{s}, \overline{t}) \geq U_1(\underline{s}, \overline{t})$, that is, $a_1^2 \geq a_3^2$. Similarly, $U_1(\overline{s}, \overline{t}') = U_1(\underline{s} \vee \overline{s}, \overline{t}')$ and quasisupermodularity implies $U_1(\underline{s} \wedge \overline{s}, \overline{t}') \geq U_1(\underline{s}, \overline{t}')$, that is, $a_2^2 \geq a_4^2$. Whenever subgame 2 is reached on the path if arbitrary profile (s, t), A_1^2 offers higher payoff to player 1 in subgame 1. Thus A_1^2 is a weakly dominant strategy in subgame 2.

Pick arbitrary $\overline{s} \in \overline{S}$ and $\underline{s} \in \underline{S}$ such that $\overline{s}^1 = \underline{s}^1$, $\overline{s}^3 = A_1^3$ and $\underline{s}^3 \neq \overline{s}^3$ and pick arbitrary $\underline{t}, \underline{t}' \in \underline{T}$ such that $\underline{t}^1 = \underline{t}'^1$, $\underline{t}^3 = B_1^3$ and $\underline{t}'^3 \neq \underline{t}^3$.

$U_1(\overline{s} \wedge \underline{s}, \underline{t}) = U_1(\underline{s}, \underline{t})$ and quasisupermodularity implies $U_1(\overline{s}, \underline{t}) \leq U_1(\underline{s} \vee \overline{s}, \underline{t})$, that is, $a_1^3 \leq a_3^3$. Similarly, $U_1(\overline{s} \wedge \underline{s}, \underline{t}') = U_1(\underline{s}, \underline{t}')$ and quasisupermodularity implies $U_1(\overline{s}, \underline{t}') \leq U_1(\underline{s} \vee \overline{s}, \underline{t}')$, that is, $a_2^3 \leq a_4^3$. Whenever subgame 3 is reached on the path if arbitrary profile (s, t), A_2^3 offers higher payoff to player 1 in subgame 3. Thus A_2^3 is a weakly dominant strategy in subgame 3.

Pick arbitrary $\overline{s} \in \overline{S}$ and $\underline{s} \in \underline{S}$ such that $\overline{s}^2 = \underline{s}^2$, $\overline{s}^4 = A_1^4$ and $\underline{s}^4 \neq \overline{s}^4$ and pick arbitrary $\overline{t}, \overline{t}' \in \overline{T}$ such that $\overline{t}^2 = \overline{t}'^2$, $\overline{t}^4 = B_1^4$ and $\overline{t}'^4 \neq \overline{t}^4$.

$U_1(\overline{s} \wedge \underline{s}, \overline{t}) = U_1(\underline{s}, \overline{t})$ and quasisupermodularity implies $U_1(\overline{s}, \overline{t}) \leq U_1(\underline{s} \vee \overline{s}, \overline{t})$, that is, $a_1^4 \leq a_3^4$. Similarly, $U_1(\overline{s} \wedge \underline{s}, \overline{t}') = U_1(\underline{s}, \overline{t}')$ and quasisupermodularity implies $U_1(\overline{s}, \overline{t}') \leq U_1(\underline{s} \vee \overline{s}, \overline{t}')$, that is, $a_2^4 \leq a_4^4$. Whenever subgame 4 is reached on the path if arbitrary profile (s, t), A_2^4 offers higher payoff to player 1 in subgame 4. Thus A_2^4 is a weakly dominant strategy in subgame 4.∎

The payoff function $U_i(s,t) : S \times T \to \mathbf{R}$ satisfies ***single crossing condition*** if for all $t, t' \in T$ such that $t \prec t'$ and for all $s, s' \in S$ such that $s \prec s'$, $u(s,t) < u(s',t)$ implies $u(s,t') < u(s',t')$ and $u(s,t) \leq u(s',t)$ implies $u(s,t') \leq u(s',t')$. The definition is equivalent to $u(s,t') \geq u(s',t')$ implies $u(s,t) \geq u(s',t)$ and $u(s,t') > u(s',t')$ implies $u(s,t) > u(s',t)$.

Claim 2. *Suppose $U_1(s,t)$ satisfies single crossing condition, the A_1^1, A_2^2, A_1^3 and A_2^4 are weakly dominant strategy in subgame 1, subgame 2, subgame 3 and subgame 4 respectively.*

Proof. Pick arbitrary $\overline{s}, \overline{s}' \in \overline{S}$ such that $\overline{s}^4 = \overline{s}'^4$, $\overline{s}^3 = A_1^3$ and $\overline{s}'^3 = A_2^3$. And pick arbitrary $\underline{t} \in \underline{T}$ and $\overline{t} \in \overline{T}$ such that $\underline{t}^3 = \overline{t}^3$.

$U_1(\overline{s} \wedge \overline{s}', \overline{t} \vee \underline{t}) = U_1(\overline{s}', \overline{t} \vee \underline{t})$ and single crossing condition implies that $U_1(\overline{s} \wedge \overline{s}', \underline{t}) \geq U_1(\overline{s}', \underline{t})$, in particular, $\underline{t}^3 = B_1^3$ implies $a_1^3 \geq a_3^3$ and $\underline{t}^3 = B_2^3$ implies $a_2^3 \geq a_4^3$. Thus whenever subgame 3 is reached on the path if arbitrary profile (s,t), A_1^3 offers higher payoff to player 1 in subgame 3. Thus A_1^3 is a weakly dominant strategy in subgame 3.

Pick arbitrary $\overline{s}, \overline{s}' \in \overline{S}$ such that $\overline{s}^3 = \overline{s}'^3$, $\overline{s}^4 = A_1^4$ and $\overline{s}'^4 = A_2^4$. And pick arbitrary $\underline{t} \in \underline{T}$ and $\overline{t} \in \overline{T}$ such that $\underline{t}^4 = \overline{t}^4$.

$U_1(\overline{s} \wedge \overline{s}', \underline{t} \wedge \overline{t}) = U_1(\overline{s}', \underline{t} \wedge \overline{t})$ and single crossing condition implies that $U_1(\overline{s} \wedge \overline{s}', \overline{t}) \geq U_1(\overline{s}', \overline{t})$, in particular, $\overline{t}^4 = B_1^4$ implies $a_1^4 \leq a_3^4$ and $\overline{t}^4 = B_2^4$ implies $a_2^4 \leq a_4^4$. Thus whenever subgame 4 is reached on the path if arbitrary profile (s, t), A_2^4 offers higher payoff to player 1 in subgame 4. Thus A_2^4 is a weakly dominant strategy in subgame 4.

Pick arbitrary $\underline{s}, \underline{s}' \in \underline{S}$ such that $\underline{s}^2 = \underline{s}'^2$, $\underline{s}^1 = A_1^1$ and $\underline{s}'^1 = A_2^1$. And pick arbitrary $\underline{t} \in \underline{T}$ and $\overline{t} \in \overline{T}$ such that $\underline{t}^1 = \overline{t}^1$.

$U_1(\underline{s} \wedge \underline{s}', \underline{t} \vee \overline{t}) = U_1(\underline{s}', \underline{t} \vee \overline{t})$ and single crossing condition implies that $U_1(\underline{s} \wedge \underline{s}', \underline{t}) \geq U_1(\underline{s}', \underline{t})$, in particular, $\underline{t}^1 = B_1^1$ implies $a_1^1 \geq a_3^1$ and $\underline{t}^1 = B_2^1$ implies $a_2^1 \geq a_4^1$. Thus whenever subgame 1 is reached on the path if arbitrary profile (s, t), A_1^1 offers higher payoff to player 1 in subgame 1. Thus A_1^1 is a weakly dominant strategy in subgame 1.

Pick arbitrary $\underline{s}, \underline{s}' \in \underline{S}$ such that $\underline{s}^1 = \underline{s}'^1$, $\underline{s}^2 = A_1^2$ and $\underline{s}'^2 = A_2^2$. And pick arbitrary $\underline{t} \in \underline{T}$ and $\overline{t} \in \overline{T}$ such that $\underline{t}^2 = \overline{t}^2$.

$U_1(\underline{s} \wedge \underline{s}', \underline{t} \wedge \overline{t}) = U_1(\underline{s}', \underline{t} \wedge \overline{t})$ and single crossing condition implies that $U_1(\underline{s} \wedge \underline{s}', \overline{t}) \leq U_1(\underline{s}', \overline{t})$, in particular, $\overline{t}^2 = B_1^2$ implies $a_1^2 \leq a_3^2$ and $\overline{t}^2 = B_2^2$ implies $a_2^2 \leq a_4^2$. Thus whenever subgame 2 is reached on the path if arbitrary profile (s, t), A_2^2 offers higher payoff to player 1 in subgame 2. Thus A_2^2 is a weakly dominant strategy in subgame 2.∎

Claim 3. *Under differential payoff to outcomes assumption, the set of two-stage game with strategic complementarities has measure zero.*

Proof. Quasisupermodularity condition requires A_1^2 to be weakly dominant in subgame 2 and A_2^3 to be weakly dominant in subgame 3. Single crossing condition requires A_2^2 to be weakly dominant in subgame 2 and A_1^3 to be weakly dominant in subgame 3. Thus $a_1^n = a_3^n$ and $a_2^n = a_4^n$ must be true for $n \in \{2, 3\}$. Contradiction to the assumption on payoff. ∎

Claim 4. *If there exist* $t \in \underline{T}$ *such that there exists* $\underline{s} \in \underline{S}$ *and* $\overline{s} \in \overline{S}$ *and* $U_1(\underline{s}, \underline{t}) < U_1(\overline{s}, \underline{t})$*, then for all* $\overline{t} \in \overline{T}$*,* $\underline{s} \in \underline{S}$ *and* $\overline{s} \in \overline{S}$*,* $U_1(\underline{s}, \overline{t}) < U_1(\overline{s}, \overline{t})$*, that is,* $max\{a_1^2, a_2^2, a_3^2, a_4^2\} < min\{a_1^4, a_2^4, a_3^4, a_4^4\}$.

Proof. Lemma 4 indicates that for every $\underline{s}' \in \{s \in \underline{S} | \ s^1 = \underline{s}^1\}$, $\overline{s}' \in \{s \in \overline{S} | \ s^3 = \overline{s}^3\}$ and $\underline{t}' \in \{t \in \underline{T} | \ t^1 = \underline{t}^1 \ and \ t^3 = \underline{t}^3\}$, $U_1(\underline{s}', \underline{t}') < U_1(\overline{s}', \underline{t}')$.

Let $\hat{t} = (B_1^0, \underline{t}^1, B_1^2, \overline{t}^3, B_1^4)$, pick arbitrary $\tilde{t} \in \{t \in \overline{T} | \ t^1 = \underline{t}^1 \ and \ t^3 = \underline{t}^3\}$, $\hat{s} \in \{s \in \underline{S} | \ s^1 = \underline{s}^1 \ and \ s^3 = \overline{s}^3\}$ and $\tilde{s} \in \{s \in \overline{S} | \ s^1 = \underline{s}^1, \ s^3 = \overline{s}^3, \ s^2 \succeq \hat{s}^2, \ s^4 \succeq \hat{s}^4\}$. Thus $\tilde{t} \succ \hat{t}$ and $\tilde{s} \succ \hat{s}$. Since $U_1(\hat{s}, \hat{t}) < U_1(\tilde{s}, \hat{t})$, single crossing condition implies $U_1(\hat{s}, \tilde{t}) < U_1(\tilde{s}, \tilde{t})$. Thus it is easy to see that all payoffs in subgame 2 is lower than payoffs in subgame 4, that is, $max\{a_1^2, a_2^2, a_3^2, a_4^2\} < min\{a_1^4, a_2^4, a_3^4, a_4^4\}$. ∎

Claim 5. *If there exists* $t \in \underline{T}$*,* $\underline{s} \in \underline{S}$ *and* $\overline{s} \in \overline{S}$ *such that* $U_1(\underline{s}, \underline{t}) < U_1(\overline{s}, \underline{t})$*. Then the following must be true:*

a *If* $\underline{s}^1 = A_1^1$*,* $\underline{t}^1 = B_1^1$*, then* $max\{a_1^1, a_2^1, a_3^1, a_4^1\} < U_1(\overline{s}, \underline{t})$.

b *If* $\underline{s}^1 = A_1^1$*,* $\underline{t}^1 = B_2^1$*, then* $max\{a_2^1, a_4^1\} < U_1(\overline{s}, \underline{t})$.

c *If* $\underline{s}^1 = A_2^1$*,* $\underline{t}^1 = B_1^1$*, then* $max\{a_3^1, a_4^1\} < U_1(\overline{s}, \underline{t})$.

d *If* $\underline{s}^1 = A_2^1$*,* $\underline{t}^1 = B_2^1$*, then* $a_4^1 < U_1(\overline{s}, \underline{t})$.

m *If* $\underline{t}^3 = B_1^3$*, then* $U_1(\underline{s}, \underline{t}) < min\{a_1^3 = a_3^3, a_2^3 = a_4^3\}$.

n *If* $\underline{t}^3 = B_2^3$*, then* $U_1(\underline{t}, \underline{t}) < a_2^3 = a_4^3$.

Proof. (a). Form $\underline{t}' = (B_1^0, B_2^1, \underline{t}^2, \underline{t}^3, \underline{t}^4)$, thus $\underline{t}' \succ \underline{t}$. Single crossing conditions require that as $U_1(\underline{s}, \underline{t}) = a_1^1 < U_1(\overline{s}, \underline{t})$, $U_1(\underline{s}, \underline{t}') = a_2^1 < U_1(\overline{s}, \underline{t}')$. As $\underline{t}'^3 = \underline{t}^3$, $U_1(\overline{s}, \underline{t}') = U_1(\overline{s}, \underline{t})$. Thus $a_2^1 < U_1(\overline{s}, \underline{t})$. Since A_1^1 is weakly dominant in subgame 1, $a_1^1 \geq a_3^1$ and $a_2^1 \geq a_4^1$. Thus $max\{a_1^1, a_2^1, a_3^1, a_4^1\} < U_1(\overline{s}, \underline{t})$.

(b). Follow directly from the assumption and A_1^1 is weakly dominant in subgame 1.

(c). Form $\underline{t}' = (B_1^0, B_2^1, \underline{t}^2, \underline{t}^3, \underline{t}^4)$, thus $\underline{t}' \succ \underline{t}$. Single crossing conditions require that as $U_1(\underline{s}, \underline{t}) = a_3^1 < U_1(\overline{s}, \underline{t})$, $U_1(\underline{s}, \underline{t}') = a_4^1 < U_1(\overline{s}, \underline{t}')$. As $\underline{t}'^3 = \underline{t}^3$, $U_1(\overline{s}, \underline{t}') = U_1(\overline{s}, \underline{t})$. Thus $a_4^1 < U_1(\overline{s}, \underline{t})$.

(d). Follow directly from the assumption.

(m). Form $\underline{t}' = (B_1^0, \underline{t}^1, \underline{t}^2, B_2^3, \underline{t}^4)$, thus $\underline{t}' \succ \underline{t}$. Single crossing conditions require that as $U_1(\underline{s}, \underline{t}) < U_1(\overline{s}, \underline{t})$, $U_1(\underline{s}, \underline{t}') < U_1(\overline{s}, \underline{t}')$, since A_1^3 and A_2^3 are both weakly dominant strategy in subgame 3, $U_1(\underline{s}, \underline{t}) < a_1^3 = a_3^3$ and $U_1(\underline{s}, \underline{t}) < a_2^3 = a_4^3$.

(n). Directly follow the assumption and $a_2^3 = a_4^3$. ∎

It is easy to see that if (a) is satisfied then (b), (c) and (d) are satisfied and moreover, both (b) and (c) imply (d). Also if (m) is satisfied, then (n) is satisfied automatically.

Now consider that there exists $\underline{t} \in \underline{T}$, $\underline{s} \in \underline{S}$ and $\overline{s} \in \overline{S}$ such that $U_1(\underline{s}, \underline{t}) < U_1(\overline{s}, \underline{t})$, then Claim (4) indicates that $max\{a_1^2, a_2^2, a_3^2, a_4^2\} < min\{a_1^4, a_2^4, a_3^4, a_4^4\}$.

Case 1: there exists $\underline{t} \in \underline{T}$ such that $a_1^1 < U_1(\overline{s}, \underline{t})$. Since (a) implies (b), (c) and (d), $max\{a_2^1, a_3^1, a_4^1\} < U_1(\overline{s}, \underline{t})$.

1. If there exists $\underline{t} \in \underline{T}$ with $\underline{t}^1 = B_1^1$ and $\underline{t}^3 = B_1^3$ and $\underline{s} \in \underline{S}$ with $\underline{s}^1 = A_1^1$ such that $U_1(\underline{s}, \underline{t}) < U_1(\overline{s}, \underline{t})$, then (m) and (a) implies $max\{a_1^1, a_2^1, a_3^1, a_4^1\} < min\{a_1^3 =$

$a^3_3, a^3_2 = a^3_4\}$.

2. If $a^1_1 \geq a^3_1 = a^3_3$ and there exists $\underline{t} \in \underline{T}$ with $\underline{t}^1 = B^1_1$ and $\underline{t}^3 = B^3_2$ and $\underline{s} \in \underline{S}$ with $\underline{s}^1 = A^1_1$ such that $U_1(\underline{s}, \underline{t}) < U_1(\overline{s}, \underline{t})$, then (a) and (n) implies $a^3_2 = a^3_4 > a^1_1 \geq a^3_1 = a^3_3$ and $max\{a^1_1, a^1_2, a^1_3, a^1_4\} < a^3_2 = a^3_4$. Thus (b), (c) and (d) must be satisfied with (n).

   (a) If there exists $\underline{s} \in \underline{S}$ with $\underline{s}^1 = A^1_1$, $\underline{t} \in \underline{T}$ with $\underline{t}^1 = B^1_2$ and $\underline{t}^3 = B^3_1$ such that $U_1(\underline{s}, \underline{t}) < U_1(\overline{s}, \underline{t})$, then (b) and (m) implies $a^3_2 = a^3_4 > a^1_1 \geq a^3_1 = a^3_3 > a^1_2 \geq a^1_4$.

      i. If there exists $\underline{s} \in \underline{S}$ with $\underline{s}^1 = A^1_2$, $\underline{t} \in \underline{T}$ with $\underline{t}^1 = B^1_1$ and $\underline{t}^3 = B^3_1$ such that $U_1(\underline{s}, \underline{t}) < U_1(\overline{s}, \underline{t})$, then (c) and (m) implies $a^3_2 = a^3_4 > a^1_1 \geq a^3_1 = a^3_3 > max\{a^1_3, a^1_2 \geq a^1_4\}$

      ii. If there exists $\underline{s} \in \underline{S}$ with $\underline{s}^1 = A^1_2$, $\underline{t} \in \underline{T}$ with $\underline{t}^1 = B^1_1$ and $\underline{t}^3 = B^3_2$ such that $U_1(\underline{s}, \underline{t}) < U_1(\overline{s}, \underline{t})$, then (c) and (n) implies $a^3_2 = a^3_4 > a^1_1 \geq a^1_3 \geq a^3_1 = a^3_3 > a^1_2 \geq a^1_4$

   (b) If there exists $\underline{s} \in \underline{S}$ with $\underline{s}^1 = A^1_1$ and $\underline{t} \in \underline{T}$ with $\underline{t}^1 = B^1_2$ and $\underline{t}^3 = B^3_2$ such that $U_1(\underline{s}, \underline{t}) < U_1(\overline{s}, \underline{t})$, then (b) and (n) implies $a^3_2 = a^3_4 > max\{a^1_1, a^1_2\} \geq min\{a^1_1, a^1_2\} \geq a^3_1 = a^3_3$

      i. If there exists $\underline{s} \in \underline{S}$ with $\underline{s}^1 = A^1_2$ and $\underline{t} \in \underline{T}$ with $\underline{t}^1 = B^1_1$ and $\underline{t}^3 = B^3_1$ such that $U_1(\underline{s}, \underline{t}) < U_1(\overline{s}, \underline{t})$, then (c) and (m) implies $a^3_2 = a^3_4 > max\{a^1_1,\ a^1_2\} > min\{a^1_1,\ a^1_2\} \geq a^3_1 = a^3_3 > max\{a^1_3, a^1_4\}$

      ii. If there exists $\underline{s} \in \underline{S}$ with $\underline{s}^1 = A^1_2$ and $\underline{t} \in \underline{T}$ with $\underline{t}^1 = B^1_1$ and $\underline{t}^3 = B^3_2$ such that $U_1(\underline{s}, \underline{t}) < U_1(\overline{s}, \underline{t})$, then (c) and (n) implies $a^3_2 = a^3_4 > a^1_1 \geq max\{a^1_3, a^1_2\} \geq min\{a^1_3, a^1_2\} \geq a^3_1 = a^3_3$

A. If there exists $\underline{s} \in \underline{S}$ with $\underline{s}^1 = A_2^1$ and $\underline{t} \in \underline{T}$ with $\underline{t}^1 = B_2^1$ and $\underline{t}^3 = B_1^3$ such that $U_1(\underline{s}, \underline{t}) < U_1(\overline{s}, \underline{t})$, then (d) and (m) implies $a_2^3 = a_4^3 > a_1^1 \geq max\{a_3^1,\ a_2^1\} \geq min\{a_3^1, a_2^1\} \geq a_1^3 = a_3^3 > a_4^1$

B. If there exists $\underline{s} \in \underline{S}$ with $\underline{s}^1 = A_2^1$ and $\underline{t} \in \underline{T}$ with $\underline{t}^1 = B_2^1$ and $\underline{t}^3 = B_2^3$ such that $U_1(\underline{s}, \underline{t}) < U_1(\overline{s}, \underline{t})$, then (d) and (n) implies that $a_2^3 = a_4^3 > a_1^1 \geq max\{a_3^1,\ a_2^1 \geq a_4^1\} \geq min\{a_3^1,\ a_2^1 \geq a_4^1\} \geq a_1^3 = a_3^3$

Payoff assignments in Case 1 are summarized as (1). $max\{a_1^1 \geq a_3^1, a_2^1 \geq a_4^1\} < min\{a_1^3 = a_3^3, a_2^3 = a_4^3\}$; (2). $a_2^3 = a_4^3 > a_1^1 \geq a_1^3 = a_3^3 > max\{a_3^1, a_2^1 \geq a_4^1\}$; (3). $a_2^3 = a_4^3 > a_1^1 \geq a_3^1 \geq a_1^3 = a_3^3 > a_2^1 \geq a_4^1$; (4). $a_2^3 = a_4^3 > max\{a_1^1,\ a_2^1\} \geq min\{a_1^1,\ a_2^1\} \geq a_1^3 = a_3^3 > max\{a_3^1, a_4^1\}$; (5). $a_2^3 = a_4^3 > a_1^1 \geq max\{a_3^1,\ a_2^1\} \geq min\{a_3^1,\ a_2^1\} \geq a_1^3 = a_3^3 > a_4^1$ and (6). $a_2^3 = a_4^3 > a_1^1 \geq max\{a_3^1,\ a_2^1 \geq a_4^1\} \geq min\{a_3^1,\ a_2^1 \geq a_4^1\} \geq a_1^3 = a_3^3$.

Case 2: $a_1^1 \geq max\{a_1^3, a_2^3, a_3^3, a_4^3\}$ and there exists $\underline{t},\ \underline{t}' \in \underline{T}$ such that $a_2^1 < U_1(\overline{s}, \underline{t})$ and $a_3^1 < U_1(\overline{s}, \underline{t}')$. Since both (b) and (c) imply (d), $a_4^1 < min\{U_1(\overline{s}, \underline{t}), U_1(\overline{s}, \underline{t}')\}$.

1. If there exists $\underline{s} \in \underline{S}$ with $\underline{s}^1 = A_1^1$ and $\underline{t} \in \underline{T}$ with $\underline{t}^1 = B_2^1$ and $\underline{t}^3 = B_1^3$ such that $U_1(\underline{s}, \underline{t}) < U_1(\overline{s}, \underline{t})$, then (b) and (m) implies that $a_1^1 \geq max\{a_1^3 = a_3^3, a_2^3 = a_4^3\} \geq min\{a_1^3 = a_3^3, a_2^3 = a_4^3\} > a_2^1 \geq a_4^1$

   (a) If there exists $\underline{s} \in \underline{S}$ with $\underline{s}^1 = A_2^1$ and $\underline{t} \in \underline{T}$ with $\underline{t}^1 = B_1^1$ and $\underline{t}^3 = B_1^3$ such that $U_1(\underline{s}, \underline{t}) < U_1(\overline{s}, \underline{t})$, then (c) and (m) implies that $a_1^1 \geq max\{a_1^3 = a_3^3, a_2^3 = a_4^3\} \geq min\{a_1^3 = a_3^3, a_2^3 = a_4^3\} > max\{a_3^1, a_2^1 \geq a_4^1\}$

   (b) If there exists $\underline{s} \in \underline{S}$ with $\underline{s}^1 = A_2^1$ and $\underline{t} \in \underline{T}$ with $\underline{t}^1 = B_1^1$ and $\underline{t}^3 = B_2^3$ such that $U_1(\underline{s}, \underline{t}) < U_1(\overline{s}, \underline{t})$, then (c) and (n) implies that $a_1^1 \geq a_2^3 = a_4^3 > a_3^1 > a_1^3 = a_3^3 > a_2^1 \geq a_4^1$.

2. If there exists $\underline{s} \in \underline{S}$ with $\underline{s}^1 = A_1^1$ and $\underline{t} \in \underline{T}$ with $\underline{t}^1 = B_2^1$ and $\underline{t}^3 = B_2^3$ such that $U_1(\underline{s}, \underline{t}) < U_1(\overline{s}, \underline{t})$, then (b) and (n) implies that $a_1^1 \geq a_2^3 = a_4^3 > a_2^1 > a_1^3 = a_3^3$.

   (a) If there exists $\underline{s} \in \underline{S}$ with $\underline{s}^1 = A_2^1$ and $\underline{t} \in \underline{T}$ with $\underline{t}^1 = B_1^1$ and $\underline{t}^3 = B_1^3$ such that $U_1(\underline{s}, \underline{t}) < U_1(\overline{s}, \underline{t})$, then (c) and (m) implies that $a_1^1 \geq a_2^3 = a_4^3 > a_2^1 > a_1^3 = a_3^3 > max\{a_3^1, a_4^1\}$.

   (b) If there exists $\underline{s} \in \underline{S}$ with $\underline{s}^1 = A_2^1$ and $\underline{t} \in \underline{T}$ with $\underline{t}^1 = B_1^1$ and $\underline{t}^3 = B_2^3$ such that $U_1(\underline{s}, \underline{t}) < U_1(\overline{s}, \underline{t})$, then (c) and (n) implies that $a_1^1 \geq a_2^3 = a_4^3 > max\{a_2^1, a_3^1\} \geq min\{a_2^1, a_3^1\} > a_1^3 = a_3^3$.

      i. If there exists $\underline{s} \in \underline{S}$ with $\underline{s}^1 = A_2^1$ and $\underline{t} \in \underline{T}$ with $\underline{t}^1 = B_2^1$ and $\underline{t}^3 = B_1^3$ such that $U_1(\underline{s}, \underline{t}) < U_1(\overline{s}, \underline{t})$, then (m) and (d) implies that $a_1^1 \geq a_2^3 = a_4^3 > max\{a_2^1, a_3^1\} \geq min\{a_2^1, a_3^1\} > a_1^3 = a_3^3 > a_4^1$.

      ii. If there exists $\underline{s} \in \underline{S}$ with $\underline{s}^1 = A_2^1$ and $\underline{t} \in \underline{T}$ with $\underline{t}^1 = B_2^1$ and $\underline{t}^3 = B_2^3$ such that $U_1(\underline{s}, \underline{t}) < U_1(\overline{s}, \underline{t})$, then (n) and (d) implies that $a_1^1 \geq a_2^3 = a_4^3 > max\{a_2^1 \geq a_4^1, a_3^1\} \geq min\{a_2^1 \geq a_4^1, a_3^1\} > a_1^3 = a_3^3$.

Payoff assignments in Case 2 are summarized as (1). $a_1^1 \geq max\{a_1^3 = a_3^3, a_2^3 = a_4^3\} \geq min\{a_1^3 = a_3^3, a_2^3 = a_4^3\} > max\{a_3^1, a_2^1 \geq a_4^1\}$; (2). $a_1^1 \geq a_2^3 = a_4^3 > a_3^1 > a_1^3 = a_3^3 > a_2^1 \geq a_4^1$; (3). $a_1^1 \geq a_2^3 = a_4^3 > a_2^1 > a_1^3 = a_3^3 > max\{a_3^1, a_4^1\}$; (4). $a_1^1 \geq a_2^3 = a_4^3 > max\{a_2^1, a_3^1\} \geq min\{a_2^1, a_3^1\} > a_1^3 = a_3^3 > a_4^1$ and (5). $a_1^1 \geq a_2^3 = a_4^3 > max\{a_2^1 \geq a_4^1, a_3^1\} \geq min\{a_2^1 \geq a_4^1, a_3^1\} > a_1^3 = a_3^3$.

Case 3: $a_1^1 \geq a_3^1 \geq max\{a_1^3, a_2^3, a_3^3, a_4^3\}$ and there exists $\underline{t} \in \underline{T}$ such that $a_2^1 < U_1(\overline{s}, \underline{t})$. Since (b) implies (d), $a_4^1 < U_1(\overline{s}, \underline{t})$.

1. If there exists $\underline{s} \in \underline{S}$ with $\underline{s}^1 = A_1^1$ and $\underline{t} \in \underline{T}$ with $\underline{t}^1 = B_2^1$ and $\underline{t}^3 = B_1^3$ such that

$U_1(\underline{s},\underline{t}) < U_1(\overline{s},\underline{t})$, then (b) and (m) implies that $a_1^1 \geq a_3^1 \geq max\{a_1^3 = a_3^3, a_2^3 = a_4^3\} \geq min\{a_1^3 = a_3^3, a_2^3 = a_4^3\} > a_2^1 \geq a_4^1$,

2. If there exists $\underline{s} \in \underline{S}$ with $\underline{s}^1 = A_1^1$ and $\underline{t} \in \underline{T}$ with $\underline{t}^1 = B_2^1$ and $\underline{t}^3 = B_2^3$ such that $U_1(\underline{s},\underline{t}) < U_1(\overline{s},\underline{t})$, then (b) and (n) implies that $a_1^1 \geq a_3^1 \geq a_2^3 = a_4^3 > a_2^1 > a_1^3 = a_3^3$.

   (a) If there exists $\underline{s} \in \underline{S}$ with $\underline{s}^1 = A_2^1$ and $\underline{t} \in \underline{T}$ with $\underline{t}^1 = B_2^1$ and $\underline{t}^3 = B_1^3$ such that $U_1(\underline{s},\underline{t}) < U_1(\overline{s},\underline{t})$, then (m) and (d) implies that $a_1^1 \geq a_3^1 \geq a_2^3 = a_4^3 > a_2^1 > a_1^3 = a_3^3 > a_4^1$

   (b) If there exists $\underline{s} \in \underline{S}$ with $\underline{s}^1 = A_2^1$ and $\underline{t} \in \underline{T}$ with $\underline{t}^1 = B_2^1$ and $\underline{t}^3 = B_2^3$ such that $U_1(\underline{s},\underline{t}) < U_1(\overline{s},\underline{t})$, then (n) and (d) implies that $a_1^1 \geq a_3^1 \geq a_2^3 = a_4^3 > a_2^1 \geq a_4^1 > a_1^3 = a_3^3$

Payoff assignments in Case 3 are summarized as (1). $a_1^1 \geq a_3^1 \geq max\{a_1^3 = a_3^3, a_2^3 = a_4^3\} \geq min\{a_1^3 = a_3^3, a_2^3 = a_4^3\} > a_2^1 \geq a_4^1$; (2). $a_1^1 \geq a_3^1 \geq a_2^3 = a_4^3 > a_2^1 > a_1^3 = a_3^3 > a_4^1$ and (3). $a_1^1 \geq a_3^1 \geq a_2^3 = a_4^3 > a_2^1 \geq a_4^1 > a_1^3 = a_3^3$.

Case 4: $min\{a_1^1, a_2^1\} \geq max\{a_1^3, a_2^3, a_3^3, a_4^3\}$ and there exists $\underline{t} \in \underline{T}$ such that $a_3^1 < U_1(\overline{s},\underline{t})$. Since (c) implies (d), $a_4^1 < U_1(\overline{s},\underline{t})$.

1. If there exists $\underline{s} \in \underline{S}$ with $\underline{s}^1 = A_2^1$ and $\underline{t} \in \underline{T}$ with $\underline{t}^1 = B_1^1$ and $\underline{t}^3 = B_1^3$ such that $U_1(\underline{s},\underline{t}) < U_1(\overline{s},\underline{t})$, then (c) and (m) implies that $min\{a_1^1,\ a_2^1\} \geq max\{a_1^3 = a_3^3,\ a_2^3 = a_4^3\} \geq min\{a_1^3 = a_3^3,\ a_2^3 = a_4^3\} > max\{a_3^1,\ a_4^1\}$

2. If there exists $\underline{s} \in \underline{S}$ with $\underline{s}^1 = A_2^1$ and $\underline{t} \in \underline{T}$ with $\underline{t}^1 = B_1^1$ and $\underline{t}^3 = B_2^3$ such that $U_1(\underline{s},\underline{t}) < U_1(\overline{s},\underline{t})$, then (c) and (n) implies that $min\{a_1^1,\ a_2^1\} \geq a_2^3 = a_4^3 > a_3^1 > a_1^3 = a_3^3$.

(a) If there exists $\underline{s} \in \underline{S}$ with $\underline{s}^1 = A_2^1$ and $\underline{t} \in \underline{T}$ with $\underline{t}^1 = B_2^1$ and $\underline{t}^3 = B_1^3$ such that $U_1(\underline{s}, \underline{t}) < U_1(\overline{s}, \underline{t})$, then (m) and (d) implies that $min\{a_1^1,\ a_2^1\} \geq a_2^3 = a_4^3 > a_3^1 > a_1^3 = a_3^3 > a_4^1$

(b) If there exists $\underline{s} \in \underline{S}$ with $\underline{s}^1 = A_2^1$ and $\underline{t} \in \underline{T}$ with $\underline{t}^1 = B_2^1$ and $\underline{t}^3 = B_2^3$ such that $U_1(\underline{s}, \underline{t}) < U_1(\overline{s}, \underline{t})$, then (n) and (d) implies that $min\{a_1^1,\ a_2^1\} \geq a_2^3 = a_4^3 > max\{a_3^1,\ a_4^1\} >= min\{a_3^1,\ a_4^1\} > a_1^3 = a_3^3$

Payoff assignments in Case 4 are summarized as (1). $min\{a_1^1,\ a_2^1\} \geq max\{a_1^3 = a_3^3,\ a_2^3 = a_4^3\} \geq min\{a_1^3 = a_3^3,\ a_2^3 = a_4^3\} > max\{a_3^1,\ a_4^1\}$; (2). $min\{a_1^1,\ a_2^1\} \geq a_2^3 = a_4^3 > a_3^1 > a_1^3 = a_3^3 > a_4^1$ and (3). $min\{a_1^1,\ a_2^1\} \geq a_2^3 = a_4^3 > max\{a_3^1,\ a_4^1\} \geq min\{a_3^1,\ a_4^1\} > a_1^3 = a_3^3$.

Case 5: $min\{a_1^1, a_2^1, a_3^1\} \geq max\{a_1^3, a_2^3, a_3^3, a_4^3\}$ and there exists $\underline{t} \in \underline{T}$ such that $a_4^1 < U_1(\overline{s}, \underline{t})$.

1. If there exists $\underline{s} \in \underline{S}$ with $\underline{s}^1 = A_2^1$ and $\underline{t} \in \underline{T}$ with $\underline{t}^1 = B_2^1$ and $\underline{t}^3 = B_1^3$ such that $U_1(\underline{s}, \underline{t}) < U_1(\overline{s}, \underline{t})$, then (m) and (d) implies that $min\{a_1^1 \geq a_3^1,\ a_2^1\} \geq max\{a_1^3 = a_3^3,\ a_2^3 = a_4^3\} \geq min\{a_1^3 = a_3^3,\ a_2^3 = a_4^3\} > a_4^1$.

2. If there exists $\underline{s} \in \underline{S}$ with $\underline{s}^1 = A_2^1$ and $\underline{t} \in \underline{T}$ with $\underline{t}^1 = B_2^1$ and $\underline{t}^3 = B_2^3$ such that $U_1(\underline{s}, \underline{t}) < U_1(\overline{s}, \underline{t})$, then (n) and (d) implies that $min\{a_1^1 \geq a_3^1,\ a_2^1\} \geq a_2^3 = a_4^3 > a_4^1 > a_1^3 = a_3^3$.

Payoff assignments in Case 5 are summarized as (1). $min\{a_1^1 \geq a_3^1,\ a_2^1\} \geq max\{a_1^3 = a_3^3,\ a_2^3 = a_4^3\} \geq min\{a_1^3 = a_3^3,\ a_2^3 = a_4^3\} > a_4^1$ and (2). $min\{a_1^1 \geq a_3^1,\ a_2^1\} \geq a_2^3 = a_4^3 > a_4^1 > a_1^3 = a_3^3$.

Claim 6. *If for every* $\underline{s} \in \underline{S}$, $\overline{s} \in \overline{S}$ *and* $\underline{t} \in \underline{T}$, $U_1(\underline{s}, \underline{t}) \geq U_1(\overline{s}, \underline{t})$ *and there exists* $\hat{t} \in \underline{T}$, $\hat{s} \in \underline{S}$ *and* $\tilde{s} \in \overline{S}$ *such that* $U_1(\hat{s}, \hat{t}) = U_1(\tilde{s}, \hat{t})$, *then for every* $\underline{s} \in \underline{S}$, $\overline{s} \in \overline{S}$ *and* $\overline{t} \in \overline{T}$, $U_1(\underline{s}, \overline{t}) \leq U_1(\overline{s}, \overline{t})$, *that is,* $max\{a_1^2, a_2^2, a_3^2, a_4^2\} \leq min\{a_1^4, a_2^4, a_3^4, a_4^4\}$.

Proof. Similar as Claim 4, Lemma 4 indicates that for every $\underline{s}' \in \{s \in \underline{S} | \ s^1 = \underline{s}^1\}$, $\overline{s}' \in \{s \in \overline{S} | \ s^3 = \overline{s}^3\}$ and $\underline{t}' \in \{t \in \underline{T} | \ t^1 = \underline{t}^1 \ and \ t^3 = \underline{t}^3\}$, $U_1(\underline{s}', \underline{t}') = U_1(\overline{s}', \underline{t}')$.

Let $\hat{t} = (B_1^0, \underline{t}^1, B_1^2, \overline{t}^3, B_1^4)$, pick arbitrary $\tilde{t} \in \{t \in \overline{T} | \ t^1 = \underline{t}^1 \ and \ t^3 = \underline{t}^3\}$, $\hat{s} \in \{s \in \underline{S} | \ s^1 = \underline{s}^1 \ and \ s^3 = \overline{s}^3\}$ and $\tilde{s} \in \{s \in \overline{S} | \ s^1 = \underline{s}^1, \ s^3 = \overline{s}^3, \ s^2 \succeq \hat{s}^2, \ s^4 \succeq \hat{s}^4\}$. Thus $\tilde{t} \succ \hat{t}$ and $\tilde{s} \succ \hat{s}$. Since $U_1(\hat{s}, \hat{t}) = U_1(\tilde{s}, \hat{t})$, single crossing condition implies $U_1(\hat{s}, \tilde{t}) \leq U_1(\tilde{s}, \tilde{t})$. Thus it is easy to see that all payoffs in subgame 2 is no higher than payoffs in subgame 4, that is, $max\{a_1^2, a_2^2, a_3^2, a_4^2\} \leq min\{a_1^4, a_2^4, a_3^4, a_4^4\}$. ∎

Claim 7. *If there exists* $t \in \underline{T}$, $\underline{s} \in \underline{S}$ *and* $\overline{s} \in \overline{S}$ *such that* $U_1(\underline{s}, \underline{t}) = U_1(\overline{s}, \underline{t})$. *Then the following must be true:*

a' *If* $\underline{s}^1 = A_1^1$, $\underline{t}^1 = B_1^1$, *then* $max\{a_1^1, a_2^1, a_3^1, a_4^1\} \leq U_1(\overline{s}, \underline{t})$.

b' *If* $\underline{s}^1 = A_1^1$, $\underline{t}^1 = B_2^1$, *then* $max\{a_2^1, a_4^1\} \leq U_1(\overline{s}, \underline{t})$.

c' *If* $\underline{s}^1 = A_2^1$, $\underline{t}^1 = B_1^1$, *then* $max\{a_3^1, a_4^1\} \leq U_1(\overline{s}, \underline{t})$.

d' *If* $\underline{s}^1 = A_2^1$, $\underline{t}^1 = B_2^1$, *then* $a_4^1 = U_1(\overline{s}, \underline{t})$.

m' *If* $\underline{t}^3 = B_1^3$, *then* $U_1(\underline{s}, \underline{t}) \leq min\{a_1^3 = a_3^3, a_2^3 = a_4^3\}$.

n' *If* $\underline{t}^3 = B_2^3$, *then* $U_1(\underline{t}, \underline{t}) \leq a_2^3 = a_4^3$.

Proof. Proved similarly as Claim 5. ∎

Now consider the scenario that for every $\underline{s} \in \underline{S}$, $\overline{s} \in \overline{S}$ and $\underline{t} \in \underline{T}$, $U_1(\underline{s}, \underline{t}) \geq U_1(\overline{s}, \underline{t})$ and there exists $\hat{t} \in \underline{T}$, $\hat{s} \in \underline{S}$ and $\tilde{s} \in \overline{S}$ such that $U_1(\hat{s}, \hat{t}) = U_1(\tilde{s}, \hat{t})$. Claim 6 indicates that $max\{a_1^2, a_2^2, a_3^2, a_4^2\} \leq min\{a_1^4, a_2^4, a_3^4, a_4^4\}$.

Case 1′: there exists $\underline{t} \in \underline{T}$ such that $a_1^1 = U_1(\overline{s}, \underline{t})$. Since (a) implies (b), (c) and (d), $max\{a_2^1, a_3^1, a_4^1\} \leq U_1(\overline{s}, \underline{t})$.

1. If there exists $\underline{t} \in \underline{T}$ with $\underline{t}^3 = B_1^3$ such that $a_1^1 = U_1(\overline{s}, \underline{t})$, then (a') and (m') imply $a_1^1 = a_3^1 = a_2^1 = a_4^1 = a_1^3 = a_3^3 = a_2^3 = a_4^3$
2. If $a_1^1 > a_1^3 = a_3^3$ and there exists $\underline{t} \in \underline{T}$ with $\underline{t}^3 = B_2^3$ such that $a_1^1 = U_1(\overline{s}, \underline{t})$, then (a') and (n') imply $a_2^3 = a_4^3 = a_1^1 = a_2^1 = a_3^1 = a_4^1 > a_1^3 = a_3^3$

Payoff assignments in case 1′ are summarized as $a_2^3 = a_4^3 = a_1^1 = a_2^1 = a_3^1 = a_4^1 \geq a_1^3 = a_3^3$.

Case 2′: $a_1^1 > max\{a_1^3, a_2^3, a_3^3, a_4^3\}$ and there exists $\underline{t} \in \underline{T}$ such that $a_2^1 = U_1(\overline{s}, \underline{t})$.

1. If there exists $\underline{t} \in \underline{T}$ with $\underline{t}^3 = B_1^3$ such that $a_2^1 = U_1(\overline{s}, \underline{t})$, then (b') and (m') imply $a_1^1 > a_2^1 = a_4^1 = a_1^3 = a_3^3 = a_2^3 = a_4^3$. If $a_3^1 = a_4^1$, then $a_1^1 > a_3^1 = a_2^1 = a_4^1 = a_1^3 = a_3^3 = a_2^3 = a_4^3$ and if $a_3^1 > a_4^1$ then $a_1^1 \geq a_3^1 > a_2^1 = a_4^1 = a_1^3 = a_3^3 = a_2^3 = a_4^3$
2. If $a_2^1 > a_1^3 = a_3^3$ and there exists $\underline{t} \in \underline{T}$ with $\underline{t}^3 = B_2^3$ such that $a_2^1 = U_1(\overline{s}, \underline{t})$, then (a') and (n') imply $a_1^1 > a_2^3 = a_4^3 = a_2^1 = a_4^1 > a_1^3 = a_3^3$. If $a_3^1 = a_4^1$, then $a_1^1 > a_2^3 = a_4^3 = a_2^1 = a_4^1 = a_3^1 > a_1^3 = a_3^3$ and if $a_3^1 > a_4^1$, then $a_1^1 \geq a_3^1 > a_2^3 = a_4^3 = a_2^1 = a_4^1 > a_1^3 = a_3^3$

Payoff assignments in case 2′ are summarized as $a_1^1 > a_2^3 = a_4^3 = a_2^1 = a_4^1 = a_3^1 \geq a_1^3 = a_3^3$ and $a_1^1 \geq a_3^1 > a_2^3 = a_4^3 = a_2^1 = a_4^1 \geq a_1^3 = a_3^3$.

Case 3′: $a_1^1, a_2^1 > max\{a_1^3, a_2^3, a_3^3, a_4^3\}$ and there exists $\underline{t} \in \underline{T}$ such that $a_3^1 = U_1(\overline{s}, \underline{t})$.

1. If there exists $\underline{t} \in \underline{T}$ with $\underline{t}^3 = B_1^3$ such that $a_3^1 = U_1(\bar{s}, \underline{t})$, then (c') and (m') imply $min\{a_1^1, a_2^1\} > a_3^1 = a_4^1 = a_1^3 = a_3^3 = a_2^3 = a_4^3$.

2. If $a_3^1 > a_1^3 = a_3^3$ and there exists $\underline{t} \in \underline{T}$ with $\underline{t}^3 = B_2^3$ such that $a_3^1 = U_1(\bar{s}, \underline{t})$, then (c') and (n') imply $min\{a_1^1, a_2^1\} > a_3^1 = a_4^1 = a_2^3 = a_4^3 > a_1^3 = a_3^3$

Payoff assignments in case $3'$ are summarized as $min\{a_1^1, a_2^1\} > a_3^1 = a_4^1 = a_2^3 = a_4^3 \geq a_1^3 = a_3^3$.

Case $4'$: $a_1^1, a_3^1 > max\{a_1^3, a_2^3, a_3^3, a_4^3\}$ and there exists $\underline{t} \in \underline{T}$ such that $a_2^1 = U_1(\bar{s}, \underline{t})$.

1. If there exists $\underline{t} \in \underline{T}$ with $\underline{t}^3 = B_1^3$ such that $a_2^1 = U_1(\bar{s}, \underline{t})$, then (b') and (m') imply $a_1^1 \geq a_3^1 > a_2^1 = a_4^1 = a_1^3 = a_3^3 = a_2^3 = a_4^3$.

2. If $a_2^1 > a_1^3 = a_3^3$ and there exists $\underline{t} \in \underline{T}$ with $\underline{t}^3 = B_2^3$ such that $a_2^1 = U_1(\bar{s}, \underline{t})$, then (b') and (n') imply $a_1^1 \geq a_3^1 > a_2^1 = a_4^1 = a_2^3 = a_4^3 > a_1^3 = a_3^3$.

Payoff assignments in case $4'$ are summarized as $a_1^1 \geq a_3^1 > a_2^1 = a_4^1 = a_2^3 = a_4^3 \geq a_1^3 = a_3^3$.

Case $5'$: $a_1^1, a_2^1, a_3^1 > max\{a_1^3, a_2^3, a_3^3, a_4^3\}$ and there exists $\underline{t} \in \underline{T}$ such that $a_4^1 = U_1(\bar{s}, \underline{t})$.

1. If there exists $\underline{t} \in \underline{T}$ with $\underline{t}^3 = B_1^3$ such that $a_4^1 = U_1(\bar{s}, \underline{t})$, then (d') and (m') imply $min\{a_1^1 \geq a_3^1, a_2^1\} > a_4^1 = a_1^3 = a_3^3 = a_2^3 = a_4^3$.

2. If $a_4^1 > a_1^3 = a_3^3$ and there exists $\underline{t} \in \underline{T}$ with $\underline{t}^3 = B_2^3$ such that $a_4^1 = U_1(\bar{s}, \underline{t})$, then (d') and (n') imply $min\{a_1^1 \geq a_3^1, a_2^1\} > a_4^1 = a_2^3 = a_4^3 > a_1^3 = a_3^3$.

Payoff assignments in case $5'$ are summarized as $min\{a_1^1 \geq a_3^1, a_2^1\} > a_4^1 = a_2^3 = a_4^3 \geq a_1^3 = a_3^3$.

Claim 8. *If there exist $\overline{t} \in \overline{T}$, $\underline{s} \in \underline{S}$ and $\overline{s} \in \overline{S}$ such that $U_1(\underline{s}, \overline{t}) > U_1(\overline{s}, \overline{t})$, then for every $\underline{t} \in \underline{T}$, $\underline{s} \in \underline{S}$ and $\overline{s} \in \overline{S}$, $U_1(\underline{s}, \underline{t}) > U_1(\overline{s}, \underline{t})$, that is, $min\{a_1^1 \geq a_3^1, a_2^1 \geq a_4^2\} > max\{a_1^3 = a_3^3, a_2^3 = a_4^3\}$.*

Proof. Proved similarly as in Claim 4. ∎

Now consider the scenario that there exists $\overline{t} \in \overline{T}$, $\underline{s} \in \underline{S}$ and $\overline{s} \in \overline{S}$ such that $U_1(\underline{s}, \overline{t}) > U_1(\overline{s}, \overline{t})$, then Claim (7) indicates that $min\{a_1^1 \geq a_3^1, a_2^1 \geq a_4^2\} > max\{a_1^3 = a_3^3, a_2^3 = a_4^3\}$.

Case 6: Suppose there exists $\overline{t} \in \overline{T}$ such that $a_4^4 < U_1(\underline{s}, \overline{t})$, payoff assignments are summarized as (1). $max\{a_3^4 \geq a_1^4, a_4^4 \geq a_2^4\} < min\{a_1^2 = a_3^2, a_2^2 = a_4^2\}$; (2). $a_1^2 = a_3^2 > a_4^4 \geq a_2^2 = a_4^2 > max\{a_2^4, a_3^4 \geq a_1^4\}$; (3). $a_1^2 = a_3^2 > a_4^4 \geq a_2^4 \geq a_2^2 = a_4^2 > a_3^4 \geq a_1^4$; (4). $a_1^2 = a_3^2 > max\{a_4^4,\ a_3^4\} \geq min\{a_4^4,\ a_3^4\} \geq a_2^2 = a_4^2 > max\{a_1^4, a_2^4\}$; (5). $a_1^2 = a_3^2 > max\{a_4^4 \geq a_2^4,\ a_3^4\} \geq min\{a_4^4 \geq a_2^4,\ a_3^4\} \geq a_2^2 = a_4^2 > a_1^4$ and (6). $a_1^2 = a_3^2 > max\{a_4^4 \geq a_2^4,\ a_3^4 \geq a_1^4\} \geq min\{a_4^4 \geq a_2^4,\ a_3^4 \geq a_1^4\} \geq a_2^2 = a_4^2$.

Case 7: Suppose $a_4^4 \geq max\{a_1^2, a_2^2, a_3^2, a_4^2\}$ and there exists $\overline{t}$, $\overline{t}' \in \overline{T}$ such that $a_3^4 < U_1(\underline{s}, \overline{t})$ and $a_2^4 < U_1(\underline{s}, \overline{t}')$, payoff assignments are summarized as (1). $a_4^4 \geq max\{a_2^2 = a_4^2, a_1^2 = a_3^2\} \geq min\{a_2^2 = a_4^2, a_1^2 = a_3^2\} > max\{a_2^4, a_3^4 \geq a_1^4\}$; (2). $a_4^4 \geq a_1^2 = a_3^2 > a_2^4 > a_2^2 = a_4^2 > a_3^4 \geq a_1^4$; (3). $a_4^4 \geq a_1^2 = a_3^2 > a_3^4 > a_2^2 = a_4^2 > max\{a_1^4, a_2^4\}$; (4). $a_4^4 \geq a_1^2 = a_3^2 > max\{a_3^4, a_2^4\} \geq min\{a_3^4, a_2^4\} > a_2^2 = a_4^2 > a_1^4$ and (5). $a_4^4 \geq a_1^2 = a_3^2 > max\{a_3^4 \geq a_1^4, a_2^4\} \geq min\{a_3^4 \geq a_1^4, a_2^4\} > a_2^2 = a_4^2$.

Case 8: Suppose $a_4^4 \geq a_2^4 \geq max\{a_1^2, a_2^2, a_3^2, a_4^2\}$ and there exists $\overline{t} \in \overline{T}$ such that $a_3^4 < U_1(\underline{s}, \overline{t})$, payoff assignments are summarized as (1). $a_4^4 \geq a_2^4 \geq max\{a_1^2 = a_3^2, a_2^2 = a_4^2\} \geq min\{a_1^2 = a_3^2, a_2^2 = a_4^2\} > a_3^4 \geq a_1^4$; (2). $a_4^4 \geq a_2^4 \geq a_1^2 = a_3^2 > a_3^4 > a_2^2 = a_4^2 > a_1^4$ and (3). $a_4^4 \geq a_2^4 \geq a_1^2 = a_3^2 > a_3^4 \geq a_1^4 > a_2^2 = a_4^2$.

Case 9: Suppose $min\{a_4^4, a_3^4\} \geq max\{a_1^2, a_2^2, a_3^2, a_4^2\}$ and there exists $\bar{t} \in \overline{T}$ such that $a_2^4 < U_1(\underline{s}, \bar{t})$, payoff assignments are summarized as (1). $min\{a_4^4,\ a_3^4\} \geq max\{a_1^2 = a_3^2,\ a_2^2 = a_4^2\} \geq min\{a_1^2 = a_3^2,\ a_2^2 = a_4^2\} > max\{a_1^4,\ a_2^4\}$; (2). $min\{a_4^4,\ a_3^4\} \geq a_1^2 = a_3^2 > a_2^4 > a_2^2 = a_4^2 > a_1^4$ and (3). $min\{a_4^4,\ a_3^4\} \geq a_1^2 = a_3^2 > max\{a_1^4,\ a_2^4\} \geq min\{a_1^4,\ a_2^4\} > a_2^2 = a_4^2$.

Case 10: Suppose $min\{a_4^4, a_3^4, a_2^4\} \geq max\{a_1^2, a_2^2, a_3^2, a_4^2\}$ and there exists $\bar{t} \in \overline{T}$ such that $a_1^4 < U_1(\underline{s}, \bar{t})$, payoff assignments are summarized as (1). $min\{a_4^4 \geq a_2^4,\ a_3^4\} \geq max\{a_1^2 = a_3^2,\ a_2^2 = a_4^2\} \geq min\{a_1^2 = a_3^2,\ a_2^2 = a_4^2\} > a_1^4$ and (2). $min\{a_4^4 \geq a_2^4,\ a_3^4\} \geq a_1^2 = a_3^2 > a_1^4 > a_2^2 = a_4^2$.

Claim 9. *If for every $\underline{s} \in \underline{S}$, $\bar{s} \in \overline{S}$ and $\bar{t} \in \overline{T}$, $U_1(\underline{s}, \bar{t}) \leq U_1(\bar{s}, \bar{t})$ and there exists $\hat{t} \in \overline{T}$, $\hat{s} \in \underline{S}$ and $\tilde{s} \in \overline{S}$ such that $U_1(\hat{s}, \hat{t}) = U_1(\tilde{s}, \hat{t})$, then $min\{a_1^1, a_2^1, a_3^1, a_4^1\} \geq max\{a_1^3, a_2^3, a_3^3, a_4^3\}$.*

Proof. Proved similarly as in Claim 6. ∎

Now consider the scenario that every $\underline{s} \in \underline{S}$, $\bar{s} \in \overline{S}$ and $\bar{t} \in \overline{T}$, $U_1(\underline{s}, \bar{t}) \leq U_1(\bar{s}, \bar{t})$ and there exists $\hat{t} \in \overline{T}$, $\hat{s} \in \underline{S}$ and $\tilde{s} \in \overline{S}$ such that $U_1(\hat{s}, \hat{t}) = U_1(\tilde{s}, \hat{t})$.

Case 6′: There exists $\bar{t} \in \overline{T}$ such that $a_4^4 = U_1(\underline{s}, \bar{t})$. Payoff assignments are summarized as $a_1^2 = a_3^2 = a_1^4 = a_2^4 = a_3^4 = a_4^4 \geq a_2^2 = a_4^2$.

Case 7′: $a_4^4 > max\{a_1^2, a_2^2, a_3^2, a_4^2\}$ and there exists $\bar{t} \in \overline{T}$ such that $a_3^4 = U_1(\underline{s}, \bar{t})$. Payoff assignments are summarized as $a_4^4 > a_1^2 = a_3^2 = a_3^4 = a_1^4 = a_2^4 \geq a_2^2 = a_4^2$ and $a_4^4 \geq a_2^4 > a_1^2 = a_3^2 = a_3^4 = a_1^4 \geq a_2^2 = a_4^2$.

Case 8′: $a_4^4, a_3^4 > max\{a_1^2, a_2^2, a_3^2, a_4^2\}$ and there exists $\bar{t} \in \overline{T}$ such that $a_2^4 = U_1(\underline{s}, \bar{t})$. Payoff assignments are summarized as $min\{a_4^4, a_3^4\} > a_2^4 = a_1^4 = a_1^2 = a_3^2 \geq a_2^2 = a_4^2$.

Case 9′: $a_4^4, a_2^4 > max\{a_1^2, a_2^2, a_3^2, a_4^2\}$ and there exists $\bar{t} \in \overline{T}$ such that $a_3^4 = U_1(\underline{s}, \bar{t})$. Payoff assignments are summarized as $a_4^4 \geq a_2^4 > a_3^4 = a_1^4 = a_1^2 = a_3^2 \geq a_2^2 = a_4^2$.

Case 10′: $a_4^4, a_3^4, a_2^4 > max\{a_1^2, a_2^2, a_3^2, a_4^2\}$ and there exists $\bar{t} \in \overline{T}$ such that $a_1^4 = U_1(\underline{s}, \bar{t})$. Payoff assignments are summarized as $min\{a_4^4 \geq a_2^4, a_3^4\} > a_1^4 = a_1^2 = a_3^2 \geq a_2^2 = a_4^2$.

Claim 10. *The two-stage game is extensive form game with strategic complementarities if one of the following is true:*

1. *Between subgame 1 and 3 are payoffs in Case* $1-5$ *and between subgame 2 and 4 is* $max\{a_2^2 = a_3^2, a_2^2 = a_4^2\} < min\{a_4^4 \geq a_2^4, a_3^4 \geq a_1^4\}$
2. *Between subgame 1 and 3 are payoffs in Case* $1'-5'$ *and* $min\{a_1^1 \geq a_3^1, a_2^1 \geq a_4^1\} > max\{a_1^3 = a_3^3, a_2^3 = a_4^3\}$ *and between subgame 2 and 4 are payoffs in Case* $6'-10'$ *and* $max\{a_2^2 = a_3^2, a_2^2 = a_4^2\} < min\{a_4^4 \geq a_2^4, a_3^4 \geq a_1^4\}$
3. *Between subgame 1 and 3 is* $min\{a_1^1 \geq a_3^1, a_2^1 \geq a_4^1\} > max\{a_1^3 = a_3^3, a_2^3 = a_4^3\}$ *and between subgame 2 and 4 are payoffs from Case* $6-10$

Proof. 1. Claim 4 indicates that if payoff are from Case $1-5$, then $max\{a_2^2 = a_3^2, a_2^2 = a_4^2\} < min\{a_4^4 \geq a_2^4, a_3^4 \geq a_1^4\}$. And $max\{a_2^2 = a_3^2, a_2^2 = a_4^2\} < min\{a_4^4 \geq a_2^4, a_3^4 \geq a_1^4\}$ allows Case $1-5$ to be satisfied under quasisupermodular and single crossing conditions.

2. If payoff are from Case $1'-5'$, then Claim 6 indicates that $max\{a_2^2 = a_3^2, a_2^2 = a_4^2\} < min\{a_4^4 \geq a_2^4, a_3^4 \geq a_1^4\}$ and moreover, under quasisupermodular and single crossing conditions requires payoff from Case $6'-10'$ or $max\{a_2^2 = a_3^2, a_2^2 = a_4^2\} < min\{a_4^4 \geq a_2^4, a_3^4 \geq a_1^4\}$ to be satisfied. And if payoff are from Case $6'-10'$, the Claim 9 indicates payoffs from Case $1'-5'$ and $min\{a_1^1 \geq a_3^1, a_2^1 \geq a_4^1\} > max\{a_1^3 = a_3^3, a_2^3 = a_4^3\}$ to be

satisfied. $min\{a_1^1 \geq a_3^1, a_2^1 \geq a_4^1\} > max\{a_1^3 = a_3^3, a_2^3 = a_4^3\}$ allows Case $6' - 10'$ and $max\{a_2^2 = a_3^2, a_2^2 = a_4^2\} < min\{a_4^4 \geq a_2^4, a_3^4 \geq a_1^4\}$ to be satisfied under quasisupermodular and single crossing conditions. And $max\{a_2^2 = a_3^2, a_2^2 = a_4^2\} < min\{a_4^4 \geq a_2^4, a_3^4 \geq a_1^4\}$ allows Case $1' - 5'$ and $min\{a_1^1 \geq a_3^1, a_2^1 \geq a_4^1\} > max\{a_1^3 = a_3^3, a_2^3 = a_4^3\}$ to be satisfied under quasisupermodular and single crossing conditions.

3. If payoffs are from Case $6 - 10$, then Claim 8 indicates $min\{a_1^1 \geq a_3^1, a_2^1 \geq a_4^1\} > max\{a_1^3 = a_3^3, a_2^3 = a_4^3\}$ to be satisfied. And $min\{a_1^1 \geq a_3^1, a_2^1 \geq a_4^1\} > max\{a_1^3 = a_3^3, a_2^3 = a_4^3\}$ allows Case $6-10$ to be satisfied under quasisupermodular and single crossing conditions. ∎

Chapter 3

3 Two-stage Games and Restricted Strategic Complements

We can easily observe that for general extensive form games, the dimensions of strategy space can be huge. This imposes considerable difficulty in analysing strategic complementarity property directly. However, this paper shows that it is possible to properly reduce the strategy space to facilitate the process without loosing generality. It is worth noting that this is a case study focusing on two-stage 2×2 games. So the results are very limited.

Mailath, Samuelson and Swinkels (1993, 1994) established that strategic independence, a property of extensive-form games and information sets can be captured in the reduced normal form. They proved a close relationship between these normal form structures and their extensive form namesakes. In particular, they developed three types of the extensive-form presentation of the normal form game. A weak representation can allow (though never force) a player to make a decision between elements in the normal form information set while knowing only his opponents have restricted themselves to a larger set. A strong representation requires that player does not choose between elements in the normal form information set until the extra information that the opponents have chosen from their normal-form information set is available. Since there exists satisfactory non-strong representations, parsimonious representation is introduced as an intermediate notion of representability that exclude information dominated sequences of actions. In particular, a weak representation is parsimonious if and only if it has no information dominance. If a game has a strong representation, then that representation is parsimonious and any parsimonious representation of the game is also a strong representation. They have also provided an algorithm that generates a representation whenever one exists. In this paper, I will implement their method and terminology.

3.1 Motivation

Let's have another look at the example we showed in Chapter 2. We have showed that the two-stage game assigned with the corresponding payoffs at terminal nodes is not an extensive form game with strategic complementarities (Echenique (2004)). However, it exhibits strategic complements as the best responses sets are increasing in strong set order with respect to opponent's strategies.

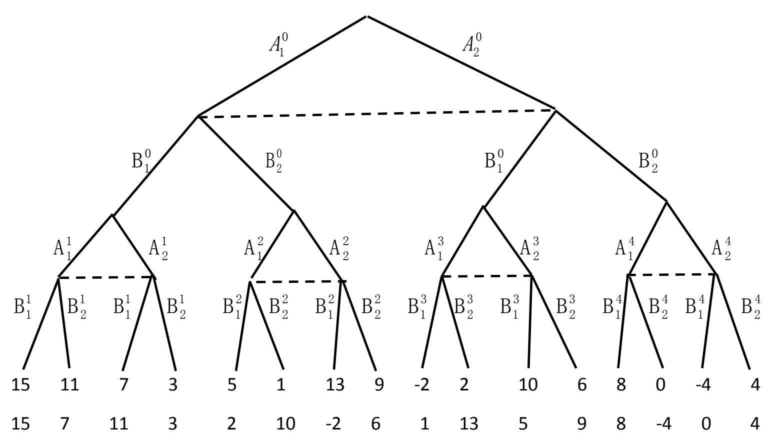

Figure 6: *Example*

Instead of the normal form of the extensive form and its subgames, we will be looking at the reduced normal form of the two-stage game.

	$s_2^{1(1,1)}$	$s_2^{1(1,2)}$	$s_2^{1(2,1)}$	$s_2^{1(2,2)}$	$s_2^{2(1,1)}$	$s_2^{2(1,2)}$	$s_2^{2(2,1)}$	$s_2^{2(2,2)}$
$s_1^{1(1,1)}$	15, 15	15, 15	11, 7	11, 7	5, 2	5, 2	1, 10	1, 10
$s_1^{1(1,2)}$	15, 15	15, 15	11, 7	11, 7	13, −2	13, −2	9, 6	9, 6
$s_1^{1(2,1)}$	7, 5	7, 5	3, 3	3, 3	5, 2	5, 2	1, 10	1, 10
$s_1^{1(2,2)}$	7, 5	7, 5	3, 3	3, 3	13, −2	13, −2	9, 6	9, 6
$s_1^{2(1,1)}$	−2, 1	2, 13	−2, 1	2, 13	8, 8	0, −4	8, 8	0, −4
$s_1^{2(1,2)}$	−2, 1	2, 13	−2, 1	2, 13	−4, 0	4, 4	−4, 0	4, 4
$s_1^{2(2,1)}$	10, 5	6, 9	10, 5	6, 9	8, 8	0, −4	8, 8	0, −4
$s_1^{2(2,2)}$	10, 5	6, 9	10, 5	6, 9	−4, 0	4, 4	−4, 0	4, 4

Figure 7: *Reduced Normal Form*

The targeted reduced-form strategies $\{s_1^{1(1,2)},\ s_1^{2(1,2)}\}$ and $\{s_2^{1(1,2)},\ s_2^{2(1,2)}\}$ were selected to resemble the strictly dominant strategy within subgames in Chapter 2. Single crossing conditions applied on the targeted reduced form strategies and no single crossing conditions were used to represent the concept of dominant strategies between subgames in Chapter 2. I showed in Section 3.5 that since the payoff assignments satisfy the conditions, the corresponding extensive form game exhibits strategic complements.

3.2 Initial Set-up

At each stage $t \in \{1,\ 2\}$, a 2-player simultaneous game is played in which player 1 and 2 is choosing from action set A_i^t, $i = 1,\ 2$. In stage t, player i's action set is $A_i^t = \{a_i^{t,\ 1},\ a_i^{t,\ 2}\}$. We assume that $a_i^{t,\ 1} \prec a_i^{t,\ 2}$. Payoffs are assigned at terminal nodes. The extensive form of the two-stage game is denoted as Γ. The set of all information sets is denoted as H,

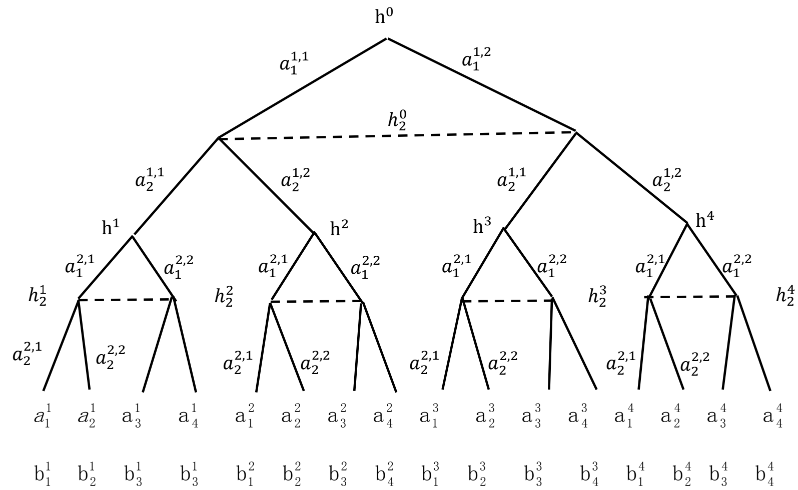

Figure 8: $2 - stage\ extensive\ form$

in particular, player 1's set of information sets is H_1 and player 2's set is H_2. Actions available on each information set h is denoted as set $A(h)$. h^0 is the initial node. We assume $h^0 \in H_1$.

Let h^1 denotes player 1's information set reached right after $a_1^{1,\ 1}$ being played at h^0 and $a_2^{1,\ 1}$ being played at h_2^0. Denote player 1's strategies that allows information set h^1 to be reached on the path as $S_1^{\Gamma}(h^1)$, player 2's strategies that reaches h_2^1 as $S_2^{\Gamma}(h_2^1)$.

Similarly, let h^2 denotes player 1's information set reached right after $a_1^{1,\ 1}$ being played at h^0 and $a_2^{1,\ 2}$ being played at h_2^0. Denote player 1's strategies that reaches h^2 as $S_1^{\Gamma}(h^2)$, player 2's strategies that reaches h_2^2 as $S_2^{\Gamma}(h_2^2)$.

Let h^3 denotes player 1's information sets reached right after $a_1^{1,\ 2}$ being played at h^0 and $a_2^{1,\ 1}$ being played at h_2^0. Denote player 1's strategies that reaches h^3 as $S_1^{\Gamma}(h^3)$, player 2's strategies that reaches h_2^3 as $S_2^{\Gamma}(h_2^3)$.

Let h^4 denotes player 1's information sets reached right after $a_1^{1,\ 2}$ being played at h^0

and $a_2^{1,\ 2}$ being played at h_2^0. Denote player 1's strategies that reaches h^4 as $S_1^\Gamma(h^4)$, player 2's strategies that reaches h_2^4 as $S_2^\Gamma(h_2^4)$.

Strategy $\hat{\sigma}_1,\ \tilde{\sigma}_1 \in S_1^\Gamma$ are **equivalent** if for every $\sigma_2 \in S_2^\Gamma$, $U_1(\hat{\sigma}_1,\ \sigma_2) = U_1(\tilde{\sigma}_1,\ \sigma_2)$. The set of all strategies that are equivalent consists of **an equivalent class**.

First, consider player 1:

Now for arbitrary $\hat{\sigma}_1,\ \tilde{\sigma}_1 \in \{\sigma_1 \in S_1^\Gamma |\ \sigma_1(h^0) = a_1^{1,\ 1},\ \sigma_1(h^1) = a_1^{2,\ 1},\ \sigma_1(h^2) = a_1^{2,\ 1}\}$ and for arbitrary $\sigma_2 \in S_2^\Gamma$, we have $U_1(\hat{\sigma}_1,\ \sigma_2) = U_1(\tilde{\sigma}_1,\ \sigma_2)$. Thus denote $s_1^{1(1,\ 1)}$ as a representation of the strategies in $\{\sigma_1 \in S_1^\Gamma |\ \sigma_1(h^0) = a_1^{1,\ 1},\ \sigma_1(h^1) = a_1^{2,\ 1},\ \sigma_1(h^2) = a_1^{2,\ 1}\}$.

Similarly, strategies in $\{\sigma_1 \in S_1^\Gamma |\ \sigma_1(h^0) = a_1^{1,\ 1},\ \sigma_1(h^1) = a_1^{2,\ i},\ \sigma_1(h^2) = a_1^{2,\ j}\}$, $i \in \{1,\ 2\}$ and $j \in \{1,\ 2\}$, yield same payoff with respect to arbitrary player 2's strategy $\sigma_2 \in S_2^\Gamma$. And denote $s_1^{1(i,\ j)}$ as a representation of the strategies in $\{\sigma_1 \in S_1^\Gamma |\ \sigma_1(h^0) = a_1^{1,\ 1},\ \sigma_1(h^1) = a_1^{2,\ i},\ \sigma_1(h^2) = a_1^{2,\ j}\}$. For arbitrary $\sigma_2 \in S_2^\Gamma$, $U_1(s_1^{1(i,\ j)},\ \sigma_2) = U_1(\hat{\sigma}_1,\ \sigma_2)$ with $\hat{\sigma}_1 \in\ s_1^{1(i,\ j)}$.

Strategies in $\{s_1^\Gamma \in S_1^\Gamma |\ s_1^\Gamma(h^0) = a_1^{1,\ 2},\ s_1^\Gamma(h^3) = a_1^{2,\ i},\ s_1^\Gamma(h^4) = a_1^{2,\ j}\}$, $i \in \{1,\ 2\}$ and $j \in \{1,\ 2\}$, yield same payoff with respect to arbitrary player 2's strategy $s_2^\Gamma \in S_2^\Gamma$. And denote $s_1^{2(i,\ j)}$ as a representation of the strategies in $\{s_1^\Gamma \in S_1^\Gamma |\ s_1^\Gamma(h^0) = a_1^{1,\ 2},\ s_1^\Gamma(h^3) = a_1^{2,\ i},\ s_1^\Gamma(h^4) = a_1^{2,\ j}\}$. For arbitrary $s_2^\Gamma \in S_2^\Gamma$, $U_1(s_1^{2(i,\ j)},\ s_2^\Gamma) = U_1(\hat{s}_1^\Gamma,\ s_2^\Gamma)$ with $\hat{s}_1^\Gamma \in\ s_1^{2(i,\ j)}$.

Reduced normal form strategy space for player 1 is $S_1 = \{s_1^{1(i,\ j)},\ s_1^{2(i,\ j)} |\ i,\ j \in \{1,\ 2\}\}$.

The order on the normal-form strategies for player 1 is assigned in the following way:

(i). For normal-form strategies within $S_1^1 = \{s_1^{1(1,\ 1)},\ s_1^{1(1,\ 2)},\ s_1^{1(2,\ 1)},\ s_1^{1(2,\ 2)}\}$, $s_1^{1(1,\ 1)} = s_1^{1(1,\ 2)} \wedge s_1^{1(2,\ 1)} \prec\ s_1^{1(1,\ 2)},\ s_1^{1(2,\ 1)} \prec\ s_1^{1(1,\ 2)} \vee s_1^{1(2,\ 1)} = s_1^{1(2,\ 2)}$.

(ii). For normal-form strategies within $S_1^2 = \{s_1^{2(1,\ 1)},\ s_1^{2(1,\ 2)},\ s_1^{2(2,\ 1)},\ s_1^{2(2,\ 2)}\}$, $s_1^{2(1,\ 1)} = s_1^{2(1,\ 2)} \wedge s_1^{2(2,\ 1)} \prec\ s_1^{2(1,\ 2)},\ s_1^{2(2,\ 1)} \prec\ s_1^{2(1,\ 2)} \vee s_1^{2(2,\ 1)} = s_1^{2(2,\ 2)}$.

(iii). The orders between elements in S_1^1 and S_1^2 is assigned in the following way: for arbitrary $s_1 \in S_1^1$ and $s_1' \in S_1^2$, $s_1 = s_1 \wedge s_1' \prec\ s_1 \vee s_1' = s_1'$.

In particular, Figure 1. showed the order on elements in S_1.

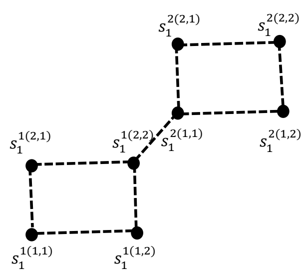

Figure 9: $order\ for\ S_1$

Now for player 2,

Denote h_2^0 as the information set of player 2 such that $S^\Gamma(h^0) = S^\Gamma(h_2^0)$.

For arbitrary $\hat{s}_2^\Gamma,\ \tilde{s}_2^\Gamma \in \{s_2^\Gamma \in S_2^\Gamma |\ s_2^\Gamma(h_2^0) = a_2^{1,\ 1},\ s_2^\Gamma(h_2^1) = a_2^{2,\ 1},\ s_2^\Gamma(h_2^3) = a_2^{2,\ 1}\}$ and for arbitrary $s_1^\Gamma \in S_1^\Gamma$, we have $U_2(s_1^\Gamma,\ \hat{s}_2^\Gamma) = U_2(s_1^\Gamma,\ \tilde{s}_2^\Gamma)$. Thus denote $s_2^{1(1,\ 1)}$ as a representation of the strategies in $\{s_2^\Gamma \in S_2^\Gamma |\ s_2^\Gamma(h_2^0) = a_2^{1,\ 1},\ s_2(h_2^1) = a_2^{2,\ 1},\ s_2(h_2^3) = a_2^{2,\ 1}\}$

Similarly, strategies in $\{s_2^\Gamma \in S_2^\Gamma |\ s_2^\Gamma(h_2^0) = a_2^{1,\ 1},\ s_2^\Gamma(h_2^1) = a_2^{2,\ i},\ s_2^\Gamma(h_2^3) = a_2^{2,\ j}\}$, $i \in \{1,\ 2\}$ and $j \in \{1,\ 2\}$, yield same payoff with respect to arbitrary player 1's strategy $s_1^\Gamma \in S_1^\Gamma$. And denote $s_2^{1(i,\ j)}$ as a representation of the strategies in $\{s_2^\Gamma \in S_2^\Gamma |\ s_2^\Gamma(h_2^0) = a_2^{1,\ 1},\ s_2^\Gamma(h_2^1) = a_2^{2,\ i},\ s_2^\Gamma(h_2^3) = a_2^{2,\ j}\}$. For arbitrary $s_1^\Gamma \in S_1^\Gamma$, $U_1(s_1^\Gamma,\ s_2^{1(i,\ j)}) = U_1(s_1^\Gamma,\ \hat{s}_2^\Gamma)$ with $\hat{s}_2^\Gamma \in\ s_2^{1(i,\ j)}$.

Strategies in $\{s_2^\Gamma \in S_2^\Gamma |\ s_2^\Gamma(h_2^0) = a_2^{1,\ 2},\ s_2^\Gamma(h_2^2) = a_2^{2,\ i},\ s_2^\Gamma(h_2^4) = a_2^{2,\ j}\}$, $i \in \{1,\ 2\}$ and $j \in \{1,\ 2\}$, yield same payoff with respect to arbitrary player 1's strategy $s_1^\Gamma \in S_1^\Gamma$. And denote $s_2^{2(i,\ j)}$ as a representation of the strategies in $\{s_2^\Gamma \in S_2^\Gamma |\ s_2^\Gamma(h_2^0) = a_2^{1,\ 2},\ s_2^\Gamma(h_2^2) = a_2^{2,\ i},\ s_2^\Gamma(h_2^4) = a_2^{2,\ j}\}$. For arbitrary $s_1^\Gamma \in S_1^\Gamma$, $U_1(s_1^\Gamma,\ s_2^{2(i,\ j)}) = U_1(s_1^\Gamma,\ \hat{s}_2^\Gamma)$ with $\hat{s}_2^\Gamma \in\ s_2^{2(i,\ j)}$.

Normal for strategy space for player 2 is $S_2 = \{s_2^{1(i,\ j)},\ s_2^{2(i,\ j)} |\ i,\ j \in \{1,\ 2\}\}$.

The order on the normal-form strategies for player 2 is assigned in the following way:

(i). For normal-form strategies within $S_2^1 = \{s_2^{1(1,\ 1)},\ s_2^{1(1,\ 2)},\ s_2^{1(2,\ 1)},\ s_2^{1(2,\ 2)}\}$, $s_2^{1(1,\ 1)} = s_2^{1(1,\ 2)} \wedge s_2^{1(2,\ 1)} \prec\ s_2^{1(1,\ 2)},\ s_2^{1(2,\ 1)} \prec\ s_2^{1(1,\ 2)} \vee s_2^{1(2,\ 1)} = s_2^{1(2,\ 2)}$.

(ii). For normal-form strategies within $S_2^2 = \{s_2^{2(1,\ 1)},\ s_2^{2(1,\ 2)},\ s_2^{2(2,\ 1)},\ s_2^{2(2,\ 2)}\}$, $s_2^{2(1,\ 1)} = s_2^{2(1,\ 2)} \wedge s_2^{2(2,\ 1)} \prec\ s_2^{2(1,\ 2)},\ s_2^{2(2,\ 1)} \prec\ s_2^{2(1,\ 2)} \vee s_2^{2(2,\ 1)} = s_2^{2(2,\ 2)}$.

(iii). The orders between elements in S_2^1 and S_2^2 is assigned in the following way: for arbitrary $s_2 \in S_2^1$ and $s_2' \in S_2^2$, $s_2 = s_2 \wedge s_2' \prec\ s_2 \vee s_2' = s_2'$.

In particular, Figure 7. showed the order on elements in S_2.

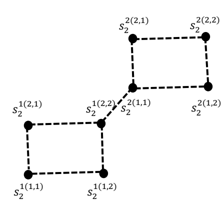

Figure 10: *order for* S_2

Figure 10 showed the reduced normal form of the two stage game with the corresponding payoff obtained from the extensive form. In particular, for arbitrary $\sigma_1 \in S_1^\Gamma$ and $\sigma_2 \in S_2^\Gamma$, there exists $s_1 \in S_1$ and $s_2 \in S_2$ such that $\sigma_1 \in s_1$ and $\sigma_2 \in s_2$ and $\pi(\sigma_1,\ \sigma_2) = \pi(s_1,\ s_2)$. Since no strategy $s_i \in S_i$ agrees with any element of $s_i \backslash \{s_i\}$ on

S_{-i}, Figure 10 is indeed the pure strategy reduced normal form of the two stage game.

	$s_2^{1(1,1)}$	$s_2^{1(1,2)}$	$s_2^{1(2,1)}$	$s_2^{1(2,2)}$	$s_2^{2(1,1)}$	$s_2^{2(1,2)}$	$s_2^{2(2,1)}$	$s_2^{2(2,2)}$
$s_1^{1(1,1)}$	a_1^1, b_1^1	a_1^1, b_1^1	a_2^1, b_2^1	a_2^1, b_2^1	a_1^2, b_1^2	a_1^2, b_1^2	a_2^2, b_2^2	a_2^2, b_2^2
$s_1^{1(1,2)}$	a_1^1, b_1^1	a_1^1, b_1^1	a_2^1, b_2^1	a_2^1, b_2^1	a_3^2, b_3^2	a_3^2, b_3^2	a_4^2, b_4^2	a_4^2, b_4^2
$s_1^{1(2,1)}$	a_3^1, b_3^1	a_3^1, b_3^1	a_4^1, b_4^1	a_4^1, b_4^1	a_1^2, b_1^2	a_1^2, b_1^2	a_2^2, b_2^2	a_2^2, b_2^2
$s_1^{1(2,2)}$	a_3^1, b_3^1	a_3^1, b_3^1	a_4^1, b_4^1	a_4^1, b_4^1	a_3^2, b_3^2	a_3^2, b_3^2	a_4^2, b_4^2	a_4^2, b_4^2
$s_1^{2(1,1)}$	a_1^3, b_1^3	a_2^3, b_2^3	a_1^3, b_1^3	a_2^3, b_2^3	a_1^4, b_1^4	a_2^4, b_2^4	a_1^4, b_1^4	a_2^4, b_2^4
$s_1^{2(1,2)}$	a_1^3, b_1^3	a_2^3, b_2^3	a_1^3, b_1^3	a_2^3, b_2^3	a_3^4, b_3^4	a_4^4, b_4^4	a_3^4, b_3^4	a_4^4, b_4^4
$s_1^{2(2,1)}$	a_3^3, b_3^3	a_4^3, b_4^3	a_3^3, b_3^3	a_4^3, b_4^3	a_1^4, b_1^4	a_2^4, b_2^4	a_1^4, b_1^4	a_2^4, b_2^4
$s_1^{2(2,2)}$	a_3^3, b_3^3	a_4^3, b_4^3	a_3^3, b_3^3	a_4^3, b_4^3	a_3^4, b_3^4	a_4^4, b_4^4	a_3^4, b_3^4	a_4^4, b_4^4

Figure 11: *Reduced Normal Form*

3.3 Main Results

Lemma 9. *Suppose single crossing condition is satisfied on $S_1^i \times S_2$, then there exists $\hat{s}_1^i \in S_1^i$ such that for arbitrary $s_2 \in S_2$ and $s_1 \in S_1^i$, $U_1(\hat{s}_1^i,\ s_2) \geq U_1(s_1,\ s_2)$, $i \in \{1,\ 2\}$.*

Proof. Notice that $s_1^{1(1,\ 1)} \prec s_1^{1(1,\ 2)}$ and $inf S_2 \prec s_2$ for arbitrary $s_2 \in S_2^2$. Since $U_1(s_1^{1(1,\ 1)},\ s_2^{1(1,\ 1)}) = U_1(s_1^{1(1,\ 2)},\ s_2^{1(1,\ 1)})$, single crossing condition implies that $U_1(s_1^{1(1,\ 1)},\ s_2) < U_1(s_1^{1(1,\ 2)},\ s_2)$ for arbitrary $s_2 \in S_2$. And for arbitrary $s_1 \in S_1^1$ and $s_2 \in S_2^2$, as $U_1(s_1^{1(1,\ 1)},\ s_2) = U_1(s_1^{1(2,\ 1)},\ s_2)$ and $U_1(s_1^{1(1,\ 2)},\ s_2) = U_1(s_1^{1(2,\ 2)},\ s_2)$, $U_1(s_1^{1(1,\ 2)},\ s_2) \geq U_1(s_1,\ s_2)$. As $s_1^{1(1,\ 1)} \prec s_1^{1(1,\ 2)}$, $s_2 \prec sup S_2$ for arbitrary $s_2 \in S_2^1$, since $U_1(s_1^{1(1,\ 1)},\ s_2^{2(2,\ 2)}) = U_1(s_1^{1(2,\ 1)},\ s_2^{2(2,\ 2)})$, single crossing condition implies that $U_1(s_1^{1(1,\ 1)},\ s_2^1) > U_1(s_1^{1(2,\ 1)},\ s_2^1)$ for arbitrary $s_2^1 \in S_2^1$. For arbitrary $s_1 \in S_1^1$ and $s_2 \in S_2^1$, as $U_1(s_1^{1(1,\ 2)},\ s_2) = U_1(s_1^{1(1,\ 1)},\ s_2)$ and $U_1(s_1^{1(2,\ 2)},\ s_2) = U_1(s_1^{1(2,\ 1)},\ s_2)$, $U_1(s_1^{1(1,\ 2)},\ s_2) > U_1(s_1,\ s_2)$. Thus for arbitrary

$s_2 \in S_2$ and $s_1' \in S_1^1$, $U_1(s_1^{1(1,\ 2)},\ s_2) \geq U_1(s_1',\ s_2)$. Similarly, for arbitrary $s_2 \in S_2$ and $s_1' \in S_1^2$, $U_1(s_1^{2(1,\ 2)},\ s_2) \geq U_1(s_1',\ s_2)$. ∎

Let $\hat{s}_1^i$ denote the strategy in the reduced strategy space S_1^i that is always included in player 1's best response set to arbitrary player 2's strategy in S_1^i. Lemma 1 showed that $\hat{s}_1^i$ always exists. And in particular, $\hat{s}_1^i = s_1^{i(1,\ 2)}$.

Function $U_1(s_1, s_2) : S_1 \times S_2 \ \rightarrow$ R satisfies **semi-no crossing conditions** between $s_1 \in S_1$ and $s_1' \in S_1$ at $\hat{s}_2 \in S_2^i$, $i \in \{1,\ 2\}$, if $U_1(s_1,\ \hat{s}_2) > U_1(s_1',\ \hat{s}_2)$, then for all $s_2 \in S_2^i$, $U_1(s_1, s_2) > U_1(s_1',\ s_2)$.

The reduced normal form game exhibits **restricted strategic complements** in player i if player i's payoff $U_i(s_i,\ s_{-i})$ is quasisupermodular on S_i, U_i satisfies single crossing conditions on $S_1 \times S_2$, and U_i satisfies semi-no crossing conditions between $s_i^{1(1,2)}$ and $s_i^{2(1,2)}$ at $\{infS_i^1,\ infS_i^2\}$.

Theorem 2. *If the reduced normal form game exhibits restricted strategic complements, then the two-stage game exhibits strategic complements.*

Proof. Let's consider player 1 here. The case for player 2 can be handled in similar ways.

Suppose the reduced normal form game exhibits restricted strategic complements in player 1, then single crossing conditions on $S_1 \times S_2$ and semi-no crossing conditions are satisfied. Lemma 9 indicates that $s_1^{1(1,2)}$ is the optimal in S_1^1, that is, $a_1^1 > a_3^1$, $a_2^1 > a_4^1$, $a_3^2 > a_1^2$ and $a_4^2 > a_2^2$. Similarly, $s_1^{2(1,2)}$ is optimal in S_1^2, that is, $a_1^3 > a_3^3$, $a_2^3 > a_4^3$, $a_3^4 > a_1^4$ and $a_4^4 > a_2^4$.

If $U_1(s_1^{1(1,2)},\ infS_2) < U_1(s_1^{2(1,2)},\ infS_2)$, then single crossing conditions on $S_1 \times S_2$ imply that $U_1(s_1^{1(1,2)},\ s_2) < U_1(s_1^{2(1,2)},\ s_2)$, for all $s_2 \in S_2$. It is easy to see that

$min\{a_1^3,\ a_2^3\} > max\{a_1^1,\ a_2^1\}$ and $min\{a_3^4, a_4^4\} > max\{a_3^2, a_4^2\}$. This result along with Lemma 9 corresponds to case (3) in Theorem 1 in which A_1^3 dominates A_2^3, A_1^1 and A_2^1, and A_2^4 dominates A_1^4, A_1^2 and A_2^2. Thus the two-stage game exhibits strategic complements.

Now consider the scenarios that $U_1(s_1^{1(1,2)},\ infS_2) > U_1(s_1^{2(1,2)},\ infS_2)$. Then semi-no-crossing condition between $s^{1(1,2)}$ and $s_1^{2(1,2)}$ on $infS_2$ indicates that for all $s_2 \in S_2^1$, $U_1(s_1^{1(1,2)},\ s_2) > U_1(s_1^{2(1,2)},\ s_2)$. Thus $min\{a_1^1,\ a_2^1\} > max\{a_1^3,\ a_2^3\}$. Along with Lemma 9, A_1^1 dominates A_2^1, A_1^3 and A_2^3.

If $U_1(s_1^{1(1,2)},\ infS_2^2) < U_1(s_1^{2(1,2)},\ infS_2^2)$, then single crossing conditions on $S_1 \times S_2$ imply that $U_1(s_1^{1(1,2)},\ s_2) < U_1(s_1^{2(1,2)},\ s_2)$, for all $s_2 \in S_2^2$. Thus $min\{a_3^4,\ a_4^4\} > max\{a_2^2,\ a_4^2\}$. With Lemma 9, A_2^4 dominates A_1^4, A_1^2 and A_2^2. Thus this corresponds to case (2) in Theorem 1. Thus the two-stage game exhibits strategic complements.

If $U_1(s_1^{1(1,2)},\ infS_2^2) > U_1(s_1^{2(1,2)},\ infS_2^2)$, then semi-no-crossing condition between $s^{1(1,2)}$ and $s_1^{2(1,2)}$ on $infS_2^2$ indicates that for all $s_2 \in S_2^2$, $U_1(s_1^{1(1,2)},\ s_2) > U_1(s_1^{2(1,2)},\ s_2)$. Thus $min\{a_3^2,\ a_4^2\} > max\{a_3^4,\ a_4^4\}$. Along with Lemma 9, A_2^2 dominates A_1^2, A_1^4 and A_2^4. This corresponds to case (1) in Theorem 1. Thus the two-stage game exhibits strategic complements. ∎

Theorem 2 indicates that standard ordinal strategic complementarity conditions imposed on reduced normal form are not sufficient to generate strategic complements in the extensive form.

If $U_1(s_1^{1(1,2)},\ infS_2) < U_1(s_1^{2(1,2)},\ infS_2)$ is satisfied, then single crossing condition imposed on $S_1 \times S_2$ alone can generate case (3) in Theorem 1. Similarly, if $U_1(s_1^{1(1,2)},\ supS_2) > U_1(s_1^{2(1,2)},\ supS_2)$, then single crossing condition alone on payoffs indicates case (1).

But without the semi-no-crossing conditions, standard ordinal strategic complementarity conditions can also result in cases where strategic complements are violated in the extensive form.

In the case that $U_1(s_1^{1(1,2)},\ infS_2) > U_1(s_1^{2(1,2)},\ infS_2)$ and $U_1(s_1^{1(1,2)},\ infS_2^2) < U_1(s_1^{2(1,2)},\ infS_2^2)$, single crossing condition can allow the scenario such that there exists $\hat{s}_2 \in S_2^1 \backslash infS_2$ and $U_1(s_1^{1(1,2)},\ \hat{s}_2) < U_1(s_1^{2(1,2)},\ \hat{s}_2)$. For example, at $s_2^{1(1,2)}$, $U_1(s_1^{1(1,2)},\ s_2^{1(1,2)}) < U_1(s_1^{2(1,2)},\ s_2^{1(1,2)})$, that is, $a_1^1 < a_2^3$, single crossing condition can be supported here. But an immediate result would be $a_1^1 > a_1^3$ and $a_1^1 < a_2^3$, thus both subgame 1 and 3 can be reached on the best response paths thus violating Lemma 1 which clearly states that only one subgame can be reached on the best response path whenever opponent's first stage action is fixed.

To see why, suppose player 2's strategy increases from $B_1^0 - B_1^1 - (B_1^2) - B_1^3 - (B_1^4)$ to $B_1^0 - B_1^1 - (B_1^2) - B_2^3 - (B_1^4)$. $a_1^1 (> a_3^1) > a_1^3 (> a_3^3)$ indicates that player 1 will choose A_1^0 then play A_1^1 in subgame 1, in particular, $s_1 = (A_1^0, A_1^1, A_1^2, A_2^3, A_1^4)$ is included in the best response set with respect to $B_1^0 - B_1^1 - (B_1^2) - B_1^3 - (B_1^4)$. $a_1^1 (> a_3^1) < a_2^3 (> a_4^3)$ indicates that player 1 will choose A_2^0 then strictly prefer to play A_1^3 in subgame 3 and indifferent among the choices in other subgames, in particular, $\hat{s}_1 = (A_2^0, A_1^1, A_1^2, A_1^3, A_1^4)$ is included in the best response set with respect to $B_1^0 - B_1^1 - (B_1^2) - B_2^3 - (B_1^4)$. But $s_1 \vee \hat{s}_1 = (A_2^0, A_1^1, A_1^2, A_2^3, A_1^4)$ is not a best response to $B_1^0 - B_1^1 - (B_1^2) - B_2^3 - (B_1^4)$, thus strategic complements is not supported here.

Semi-no crossing condition between $s_1^{1(1,2)}$ and $s_1^{2(1,2)}$ on $infS_2^1$ ensures that such scenarios will not arise. And in particular, enables case (2).

Now we move on to find a characterization of strategic complements on two-stage games.

Function $U_1(s_1, s_2) : S_1 \times S_2 \ \rightarrow$ R satisfies **no crossing conditions** between $\hat{s}_1 \in S_1^i$ and $\tilde{s}_1 \in S_1^j$ at $\hat{s}_2 \in S_2^k$, $i,\ j,\ k \in \{1,\ 2\}$, if $U_1(\hat{s}_1,\ \hat{s}_2) > U_1(\tilde{s}_1,\ \hat{s}_2)$, then for all $s_1 \in S_1$ and $s_2 \in S_2^k$, $U_1(\hat{s}_1, s_2) \geq\ U_1(s_1,\ s_2)$.

Lemma 10. *If player 1's payoff satisfies single crossing conditions on* $\{s_1^{1(1,2)},\ s_1^{2(1,2)}\} \times \{s_2^{1(1,2)},\ s_2^{2(1,2)}\}$ *and no crossing conditions between* $s_1^{1(1,2)}$ *and* $s_1^{2(1,2)}$ *at* $\{s_2^{1(1,2)},\ s_2^{2(1,2)}\}$, *then the reduced normal form game exhibits strategic complements.*

Proof. Player 1's payoff satisfies single crossing condition in $\{\hat{s}_1^1,\ \hat{s}_1^2\} \times \{\hat{s}_2^1,\ \hat{s}_2^2\}$, thus there exists three scenarios. In scenario 1, $U_1(\hat{s}_1^1,\ \hat{s}_2^1) < U_1(\hat{s}_1^2,\ \hat{s}_2^1)$ implies $U_1(\hat{s}_1^1,\ \hat{s}_2^2) < U_1(\hat{s}_1^2,\ \hat{s}_2^2)$; in scenario 2, $U_1(\hat{s}_1^1,\ \hat{s}_2^2) > U_1(\hat{s}_1^2,\ \hat{s}_2^2)$ implies $U_1(\hat{s}_1^1,\ \hat{s}_2^1) > U_1(\hat{s}_1^2,\ \hat{s}_2^1)$ and in scenario 3, $U_1(\hat{s}_1^1,\ \hat{s}_2^1) > U_1(\hat{s}_1^2,\ \hat{s}_2^1)$ and $U_1(\hat{s}_1^1,\ \hat{s}_2^2) < U_1(\hat{s}_1^2,\ \hat{s}_2^2)$.

In scenario 1, as $U_1(\hat{s}_1^1,\ \hat{s}_2^1) < U_1(\hat{s}_1^2,\ \hat{s}_2^1)$ and $\hat{s}_2^1 \in S_2^1$, no crossing conditions in $S_1 \times S_2$ imply that for arbitrary $s_2 \in S_2^1$ and $s_1 \in S_1 \backslash \hat{s}_1^2$, $U_1(s_1,\ s_2) \leq U_1(\hat{s}_1^2,\ s_2)$. Similarly, as $U_1(\hat{s}_1^1,\ \hat{s}_2^2) < U_1(\hat{s}_1^2,\ \hat{s}_2^2)$ and $\hat{s}_2^2 \in S_2^2$, no crossing conditions implies that for arbitrary $s_2 \in S_2^2$ and for arbitrary $s_1 \in S_1 \backslash \hat{s}_1^2$, $U_1(s_1,\ s_2) \leq U_1(\hat{s}_1^2,\ s_2)$. Thus for arbitrary $s_2 \in S_2^1$, $BR^1(s_2) = \{s_1^{2(1,\ 1)},\ s_1^{2(1,\ 2)}\}$ and for arbitrary $s_2 \in S_2^2$, $BR^1(s_2) = \{s_1^{2(1,\ 2)},\ s_1^{2(2,\ 2)}\}$. Thus this normal form exhibits strategic complementarities.

In scenario 2, as $U_1(\hat{s}_1^1,\ \hat{s}_2^2) > U_1(\hat{s}_1^2,\ \hat{s}_2^2)$ and $\hat{s}_2^2 \in S_2^2$, no crossing conditions in $S_1 \times S_2$ imply that for arbitrary $s_2 \in S_2^2$ and $s_1 \in S_1 \backslash \hat{s}_1^1$, $U_1(\hat{s}_1^1,\ s_2) \geq U_1(s_1,\ s_2)$. Similarly, as $U_1(\hat{s}_1^1,\ \hat{s}_2^1) > U_1(\hat{s}_1^2,\ \hat{s}_2^1)$ and $\hat{s}_2^1 \in S_2^1$, no crossing conditions imply that for arbitrary $s_2 \in S_2^1$ and for arbitrary $s_1 \in S_1 \backslash \hat{s}_1^1$, $U_1(\hat{s}_1^1,\ s_2) \geq U_1(s_1,\ s_2)$. Thus for arbitrary $s_2 \in S_2^1$, $BR^1(s_2) = \{s_1^{1(1,\ 1)},\ s_1^{1(1,\ 2)}\}$ and for arbitrary $s_2 \in S_2^2$, $BR^1(s_2) = \{s_1^{1(1,\ 2)},\ s_1^{1(2,\ 2)}\}$. Thus this normal form exhibits strategic complementarities.

In scenario 3, as $U_1(\hat{s}_1^1,\ \hat{s}_2^1) > U_1(\hat{s}_1^2,\ \hat{s}_2^1)$ and $\hat{s}_2^1 \in S_2^1$, no crossing conditions imply that for arbitrary $s_2 \in S_2^1$ and $s_1 \in S_1 \backslash \hat{s}_1^1$, $U_1(\hat{s}_1^1,\ s_2) \geq U_1(s_1,\ s_2)$. Similarly, as $U_1(\hat{s}_1^1,\ \hat{s}_2^2) < U_1(\hat{s}_1^2,\ \hat{s}_2^2)$ and $\hat{s}_2^2 \in S_2^2$, no crossing conditions imply that for arbitrary $s_2 \in S_2^2$ and $s_1 \in S_1 \backslash \hat{s}_1^2$, $U_1(s_1,\ s_2) < U_1(\hat{s}_1^2,\ s_2)$. Thus $BR^1(s_2) = \{s_1^{1(1,\ 1)},\ s_1^{1(1,\ 2)}\}$ for $s_2 \in S_2^1$ and $BR^1(s_2) = \{s_1^{2(1,\ 2)},\ s_1^{2(2,\ 2)}\}$ for $s_2 \in S_2^2$. For arbitrary $\hat{s}_2 \in S_2^1$ and arbitrary $\tilde{s}_2 \in S_2^2$, $\hat{s}_2 \prec \tilde{s}_2$, since $BR^1(\hat{s}_2) \sqsubseteq BR^1(\tilde{s}_2)$, this normal form game exhibits strategic complementarities.

Lemma 11. *Player 1's payoff satisfies single crossing conditions on* $\{s_1^{1(1,2)},\ s_1^{2(1,2)}\} \times \{s_2^{1(1,2)},\ s_2^{2(1,2)}\}$ *and no crossing conditions between* $s_1^{1(1,2)}$ *and* $s_1^{2(1,2)}$ *at* $\{s_2^{1(1,2)},\ s_2^{2(1,2)}\}$ *if and only if one of the three conditions is satisfied*

(1) $a_1^3 > a_3^3,\ a_2^3 > a_4^3,\ min\{a_1^3,\ a_2^3\} > \max\{a_1^1,\ a_2^1,\ a_3^1,\ a_4^1\}$ *and* $a_3^4 > a_1^4,\ a_4^4 > a_2^4,\ min\{a_3^4,\ a_4^4\} > max\{a_1^2,\ a_2^2,\ a_3^2,\ a_4^2\}$*;*

(2) $a_1^1 > a_3^1,\ a_2^1 > a_4^1,\ min\{a_1^1,\ a_2^1\} > \max\{a_1^3,\ a_2^3,\ a_3^3,\ a_4^3\}$ *and* $a_3^2 > a_1^2,\ a_4^2 > a_2^2,\ min\{a_3^2,\ a_4^2\} > max\{a_1^4,\ a_2^4,\ a_3^4,\ a_4^4\}$*;*

(3) $a_1^1 > a_3^1,\ a_2^1 > a_4^1,\ min\{a_1^1,\ a_2^1\} > \max\{a_1^3,\ a_2^3,\ a_3^3,\ a_4^3\}$ *and* $a_3^4 > a_1^4,\ {}_4^4 > a_2^4,\ min\{a_3^4,\ a_4^4\} > max\{a_1^2,\ a_2^2,\ a_3^2,\ a_4^2\}$.

Proof. Suppose the reduced normal form game exhibits restricted strategic complementarities, then the payoff assignments in scenario 1 are $a_1^3 > a_3^3$ and $a_3^3 > max\{a_1^1,\ a_2^1,\ a_3^1,\ a_4^1\}$, $a_2^3 > a_4^3$ and $a_2^3 > max\{a_1^1,\ a_2^1,\ a_3^1,\ a_4^1\}$, $a_3^4 > a_1^4$ and $a_3^4 > max\{a_1^2,\ a_2^2,\ a_3^2,\ a_4^2\}$, $a_4^4 > a_2^4$ and $a_4^4 > max\{a_1^2,\ a_2^2,\ a_3^2,\ a_4^2\}$.

The payoff assignments in scenario 2 are $a_1^1 > a_3^1$ and $a_1^1 > max\{a_1^3,\ a_2^3,\ a_3^3,\ a_4^3\}$, $a_2^1 > a_4^1$ and $a_2^1 > max\{a_1^3,\ a_2^3,\ a_3^3,\ a_4^3\}$, $a_3^2 > a_1^2$ and $a_3^2 > max\{a_1^4,\ a_2^4,\ a_3^4,\ a_4^4\}$ and

$a_4^2 > a_2^2$ and $a_4^2 > max\{a_1^4,\ a_2^4,\ a_3^4,\ a_4^4\}$.

The payoff assignments in scenario 3 are $a_1^1 > a_3^1$ and $a_1^1 > max\{a_1^3,\ a_2^3,\ a_3^3,\ a_4^3\}$, $a_2^1 > a_4^1$ and $a_2^1 > max\{a_1^3,\ a_2^3,\ a_3^3,\ a_4^3\}$ and $a_3^4 > a_1^4$ and $a_3^4 > max\{a_1^2,\ a_2^2,\ a_3^2,\ a_4^2\}$, $a_2^4 > a_4^4$ and $a_2^4 > max\{a_1^2,\ a_2^2,\ a_3^2,\ a_4^2\}$.

If one of the following three conditions are satisfied, then it is easy to check that both single crossing condition and no crossing conditions are satisfied.

Suppose (1) is satisfied, then single crossing condition $\{\hat{s}_1^1,\ \hat{s}_1^2\} \times \{\hat{s}_2^1,\ \hat{s}_2^2\}$ is satisfied as $a_2^3 > a_1^1$ and $a_4^4 > a_3^2$. No crossing condition is satisfied as for arbitrary $s_2 \in S_2^1$ and $s_1 \in S_1 \backslash \hat{s}_1^2$, $U_1(s_1,\ s_2) \leq U_1(\hat{s}_1^2,\ s_2)$ and for arbitrary $s_2 \in S_2^2$ and $s_1 \in S_1 \backslash \hat{s}_1^2$, $U_1(s_1,\ s_2) \leq U_1(\hat{s}_1^2,\ s_2)$.

Suppose (2) is satisfied, then single crossing condition $\{\hat{s}_1^1,\ \hat{s}_1^2\} \times \{\hat{s}_2^1,\ \hat{s}_2^2\}$ is satisfied as $a_3^2 > a_4^4$ and $a_1^1 > a_2^3$. No crossing condition is satisfied as for arbitrary $s_2 \in S_2^1$ and $s_1 \in S_1 \backslash \hat{s}_1^2$, $U_1(s_1,\ s_2) \leq U_1(\hat{s}_1^2,\ s_2)$ and for arbitrary $s_2 \in S_2^2$ and $s_1 \in S_1 \backslash \hat{s}_1^2$, $U_1(s_1,\ s_2) \leq U_1(\hat{s}_1^2,\ s_2)$.

Suppose (3) is satisfied, then single crossing condition $\{\hat{s}_1^1,\ \hat{s}_1^2\} \times \{\hat{s}_2^1,\ \hat{s}_2^2\}$ is satisfied as $a_1^1 > a_2^3$ and $a_3^2 < a_4^4$. No crossing condition is satisfied as for arbitrary $s_2 \in S_2^1$ and $s_1 \in S_1 \backslash \hat{s}_1^2$, $U_1(s_1,\ s_2) \leq U_1(\hat{s}_1^2,\ s_2)$ and for arbitrary $s_2 \in S_2^2$ and $s_1 \in S_1 \backslash \hat{s}_1^2$, $U_1(s_1,\ s_2) \leq U_1(\hat{s}_1^2,\ s_2)$.

Thus the reduced normal form game with those payoffs exhibit restricted strategic complementarities. ∎

Theorem 3. *Under differential payoff to outcome, the following claims are equivalent:*

i. player 1's payoffs satisfies single crossing conditions on $\{s_1^{1(1,2)},\ s_1^{2(1,2)}\} \times \{s_2^{1(1,2)},\ s_2^{2(1,2)}\}$

and no crossing conditions between $s_1^{1(1,2)}$ and $s_1^{2(1,2)}$ at $\{s_2^{1(1,2)},\ s_2^{2(1,2)}\}$

ii. the two-stage game exhibits strategic complements in player 1.

Proof. We can invoke the Theorem 1 in chapter 2 and Lemma 11 to prove the equivalence between those two claims.

Corollary 5. *For every best response sets in the reduced normal form game with restricted strategic complementarities, their extensive form correspondence are also best response sets in a two-stage game with strategic complementarities.*

Corollary 6. *For every best response sets in a two-stage game with strategic complementarities, their reduced normal form correspondence are also best response sets in a reduced normal form game with restricted strategic complementarities.*

Lemma 2 indicates that there are three possible best responses for the reduced normal form game with restricted strategic complementarities.

In the first case, $BR^1(s_2) = \{s_1^{2(1,\ 1)},\ s_1^{2(1,\ 2)}\}$ for $s_2 \in S_2^1$ and $BR^1(s_2) = \{s_1^{2(1,\ 2)},\ s_1^{2(2,\ 2)}\}$ for $s_2 \in S_2^2$. The corresponding best response sets in the extensive form is the following: $BR^1(s_2^\Gamma) = s_1^{2(1,\ 1)} \cup\ s_1^{2(1,\ 2)} = \{s_1^\Gamma \in S_1^\Gamma |\ s_1^\Gamma(h^0) = a_1^{1,\ 2},\ s_1^\Gamma(h^3) = a_1^{2,\ 1}\}$ for $s_2^\Gamma \in \{s_2^\Gamma \in S_2^\Gamma |\ s_2^\Gamma(h_2^0) = a_2^{1,\ 1}\}$ and $BR^1(s_2^\Gamma) = s_1^{2(1,\ 2)} \cup\ s_1^{2(2,\ 2)} = \{s_1^\Gamma \in S_1^\Gamma |\ s_1^\Gamma(h^0) = a_1^{1,\ 2},\ s_1^\Gamma(h^4) = a_1^{2,\ 2}\}$ for $s_2^\Gamma \in \{s_2^\Gamma \in S_2^\Gamma |\ s_2^\Gamma(h_2^0) = a_2^{1,\ 2}\}$

In this case, $BR^1(s_2^\Gamma)$ are complete sublattices for $s_2^\Gamma \in \{s_2^\Gamma \in S_2^\Gamma |\ s_2^\Gamma(h_2^0) = a_2^{1,\ 1}\}$. Similarly, $BR^1(s_2^\Gamma)$ are complete sublattices for $s_2^\Gamma \in \{s_2^\Gamma \in S_2^\Gamma |\ s_2^\Gamma(h_2^0) = a_2^{1,\ 2}\}$. Pick arbitrary $\hat{s}_2^\Gamma \in \{s_2^\Gamma \in S_2^\Gamma |\ s_2^\Gamma(h_2^0) = a_2^{1,\ 1}\}$ and $\tilde{s}_2^\Gamma \in \{s_2^\Gamma \in S_2^\Gamma |\ s_2^\Gamma(h_2^0) = a_2^{1,\ 2}\}$ such that $\hat{s}_2^\Gamma \prec \tilde{s}_2^\Gamma$, pick arbitrary $\hat{s}_1^\Gamma \in BR^1(\hat{s}_2^\Gamma)$ and $\tilde{s}_1^\Gamma \in BR^1(\tilde{s}_2^\Gamma)$, $\hat{s}_2^\Gamma \wedge \tilde{s}_2^\Gamma \in BR^1(\hat{s}_2^\Gamma)$ and

$\hat{s}_2^\Gamma \vee \tilde{s}_2^\Gamma \in BR^1(\tilde{s}_2^\Gamma)$. Thus for extensive form games with such best response sets, they exhibits strategic complementarities.

In the second case, $BR^1(s_2) = \{s_1^{1(1,\ 1)},\ s_1^{1(1,\ 2)}\}$ for $s_2 \in S_2^1$ and $BR^1(s_2) = \{s_1^{1(1,\ 2)},\ s_1^{1(2,\ 2)}\}$ for $s_2 \in S_2^2$. The corresponding best response sets in the extensive form is the following: $BR^1(s_2^\Gamma) = s_1^{1(1,\ 1)} \cup\ s_1^{1(1,\ 2)} = \{s_1^\Gamma \in S_1^\Gamma |\ s_1^\Gamma(h^0) = a_1^{1,\ 1},\ s_1^\Gamma(h^1) = a_1^{2,\ 1}\}$ for $s_2^\Gamma \in \{s_2^\Gamma \in S_2^\Gamma |\ s_2^\Gamma(h_2^0) = a_2^{1,\ 1}\}$ and $BR^1(s_2^\Gamma) = s_1^{1(1,\ 2)} \cup\ s_1^{1(2,\ 2)} = \{s_1^\Gamma \in S_1^\Gamma |\ s_1^\Gamma(h^0) = a_1^{1,\ 1},\ s_1^\Gamma(h^2) = a_1^{2,\ 2}\}$ for $s_2^\Gamma \in \{s_2^\Gamma \in S_2^\Gamma |\ s_2^\Gamma(h_2^0) = a_2^{1,\ 2}\}$.

In this case, $BR^1(s_2^\Gamma)$ are complete sublattices for all $s_2^\Gamma \in \{s_2^\Gamma \in S_2^\Gamma |\ s_2^\Gamma(h_2^0) = a_2^{1,\ 1}\}$. Similarly, $BR^1(s_2^\Gamma)$ are complete sublattices for all $s_2^\Gamma \in \{s_2^\Gamma \in S_2^\Gamma |\ s_2^\Gamma(h_2^0) = a_2^{1,\ 2}\}$. Pick arbitrary $\hat{s}_2^\Gamma \in \{s_2^\Gamma \in S_2^\Gamma |\ s_2^\Gamma(h_2^0) = a_2^{1,\ 1}\}$ and $\tilde{s}_2^\Gamma \in \{s_2^\Gamma \in S_2^\Gamma |\ s_2^\Gamma(h_2^0) = a_2^{1,\ 2}\}$ such that $\hat{s}_2^\Gamma \prec \tilde{s}_2^\Gamma$, $BR^1(\hat{s}_2^\Gamma) \sqsubseteq BR^1(\tilde{s}_2^\Gamma)$. Thus for extensive form games with such best response sets, they exhibits strategic complementarities.

In the third case, $BR^1(s_2) = \{s_1^{1(1,\ 1)},\ s_1^{1(1,\ 2)}\}$ for $s_2 \in S_2^1$ and $BR^1(s_2) = \{s_1^{2(1,\ 2)},\ s_1^{2(2,\ 2)}\}$ for $s_2 \in S_2^2$. The corresponding best response sets in the extensive form is the following: $BR^1(s_2^\Gamma) = s_1^{1(1,\ 1)} \cup\ s_1^{1(1,\ 2)} = \{s_1^\Gamma \in S_1^\Gamma |\ s_1^\Gamma(h^0) = a_1^{1,\ 1},\ s_1^\Gamma(h^1) = a_1^{2,\ 1}\}$ for $s_2^\Gamma \in \{s_2^\Gamma \in S_2^\Gamma |\ s_2^\Gamma(h_2^0) = a_2^{1,\ 1}\}$ and $BR^1(s_2^\Gamma) = s_1^{2(1,\ 2)} \cup\ s_1^{2(2,\ 2)} = \{s_1^\Gamma \in S_1^\Gamma |\ s_1^\Gamma(h^0) = a_1^{1,\ 2},\ s_1^\Gamma(h^4) = a_1^{2,\ 2}\}$ for $s_2^\Gamma \in \{s_2^\Gamma \in S_2^\Gamma |\ s_2^\Gamma(h_2^0) = a_2^{1,\ 2}\}$.

In this case, $BR^1(s_2^\Gamma)$ are complete sublattices for $s_2^\Gamma \in \{s_2^\Gamma \in S_2^\Gamma |\ s_2^\Gamma(h_2^0) = a_2^{1,\ 1}\}$. Similarly, $BR^1(s_2^\Gamma)$ are complete sublattices for $s_2^\Gamma \in \{s_2^\Gamma \in S_2^\Gamma |\ s_2^\Gamma(h_2^0) = a_2^{1,\ 2}\}$. Pick arbitrary $\hat{s}_2^\Gamma \in \{s_2^\Gamma \in S_2^\Gamma |\ s_2^\Gamma(h_2^0) = a_2^{1,\ 1}\}$ and $\tilde{s}_2^\Gamma \in \{s_2^\Gamma \in S_2^\Gamma |\ s_2^\Gamma(h_2^0) = a_2^{1,\ 2}\}$ such that $\hat{s}_2^\Gamma \prec \tilde{s}_2^\Gamma$, pick arbitrary $\hat{s}_1^\Gamma \in BR^1(\hat{s}_2^\Gamma)$ and $\tilde{s}_1^\Gamma \in BR^1(\tilde{s}_2^\Gamma)$, $\hat{s}_2^\Gamma \wedge \tilde{s}_2^\Gamma \in BR^1(\hat{s}_2^\Gamma)$ and $\hat{s}_2^\Gamma \vee \tilde{s}_2^\Gamma \in BR^1(\tilde{s}_2^\Gamma)$. Thus for extensive form games with such best response sets, they exhibits strategic complements.

I would like to point out that ordinal complementarity conditions imposed on the reduced normal form are not sufficient to generate strategic complements in the extensive form. We will show that standard ordinal complementarity conditions (single crossing condition and quasi-supermodularity condition) and semi-no crossing conditions between $s_1^{1(1,\ 2)}$ and $s_1^{2(1,\ 2)}$ at $\{infS_2^1,\ infS_2^2\}$ imposed on reduced normal form are sufficient to generate strategic complements in the extensive form.

Single crossing conditions alone can generate case (1) and case (3) in Theorem 1. Semi-no crossing conditions serves as a brutal force to enable case (2). Notice that, quasi-supermodularity condition will not impose any additional payoff restrictions here, thus is conveniently left out.

First of all, if $U_1(s_1^{1(1,2)},\ infS_2) < U_1(s_1^{2(1,2)},\ infS_2)$, then single crossing conditions on $S_1 \times S_2$ imply that $U_1(s_1^{1(1,2)},\ s_2) < U_1(s_1^{2(1,2)},\ s_2)$, for all $s_2 \in S_2$. Along with Lemma 9, it is easy to check that case (1) applies.

Secondly, if $U_1(s_1^{1(1,2)},\ supS_2) > U_1(s_1^{2(1,2)},\ supS_2)$, then the contrapositive part of single crossing conditions on $S_1 \times S_2$ imply that $U_1(s_1^{1(1,2)},\ s_2) > U_1(s_1^{2(1,2)},\ s_2)$, for all $s_2 \in S_2$. Along with Lemma 9, it is easy to check that case (3) applies.

In the case that $U_1(s_1^{1(1,2)},\ infS_2) > U_1(s_1^{2(1,2)},\ infS_2)$ and $U_1(s_1^{1(1,2)},\ infS_2^2) < U_1(s_1^{2(1,2)},\ infS_2^2)$, single crossing condition can allow the scenario such that there exists $\hat{s}_2 \in S_2^1 \backslash infS_2$ such that $U_1(s_1^{1(1,2)},\ \hat{s}_2) < U_1(s_1^{2(1,2)},\ \hat{s}_2)$. Thus violating Lemma 1 in Chapter 2. Semi-no crossing condition between $s_1^{1(1,2)}$ and $s_1^{2(1,2)}$ on $infS_2^1$ ensures that such scenarios will not arise. And in particular, enables case (2).

Similarly, in the case that $U_1(s_1^{1(1,2)},\ supS_2^1) > U_1(s_1^{2(1,2)},\ supS_2^1)$ and $U_1(s_1^{1(1,2)},\ infS_2^2) >$

$U_1(s_1^{2(1,2)},\ infS_2^2)$, semi-no crossing condition between $s_1^{1(1,2)}$ and $s_1^{2(1,2)}$ on $infS_2^2$ ensures that case (1) will be satisfied.

3.4 Relating to MSS's Work

For the purpose of current research, it worth noting that any extensive form game with distinct terminal payoffs is necessarily a strong representation of the corresponding reduced normal form. With the stage game structure, their algorithm will recover the exact same extensive form as the original two-stage games and it is a satisfying representation of the normal form. Thus their research provided a sound theoretical foundation for us to explore the reduced normal form of the normal form of the two-stage 2×2 games.

MSS(1994) showed that for extensive form game with distinct payoff at each terminal nodes, it is a strong presentation of the corresponding PRNF (Theorem 9) and a representation algorithm applied on such PRNF can find a strong representation (Theorem 8). I will show that the extensive form generated by the algorithm turn out to have the exact tree structure as the original extensive form with differences in the way actions are indexed.

Thus we are comfortable to say that by turning to reduced normal form, we will not loose any strategic relevant aspect of a game.

For this game, the relevant information sets for both player 1 and 2 are the same, that are:

S

$X^1 = S_1^1 \times\ S_2^1 = \{s_1^{1(1,\ 1)},\ s_1^{1(1,\ 2)},\ s_1^{1(2,\ 1)},\ s_1^{1(2,\ 2)}\} \times\ \{s_2^{1(1,\ 1)},\ s_2^{1(1,\ 2)},\ s_2^{1(2,\ 1)},\ s_2^{1(2,\ 2)}\}$

$X^2 = S_1^1 \times\ S_2^2 = \{s_1^{1(1,\ 1)},\ s_1^{1(1,\ 2)},\ s_1^{1(2,\ 1)},\ s_1^{1(2,\ 2)}\} \times\ \{s_2^{2(1,\ 1)},\ s_2^{2(1,\ 2)},\ s_2^{2(2,\ 1)},\ s_2^{2(2,\ 2)}\}$

$X^3 = S_1^2 \times\ S_2^1 = \{s_1^{2(1,\ 1)},\ s_1^{2(1,\ 2)},\ s_1^{2(2,\ 1)},\ s_1^{2(2,\ 2)}\} \times\ \{s_2^{1(1,\ 1)},\ s_2^{1(1,\ 2)},\ s_2^{1(2,\ 1)},\ s_2^{1(2,\ 2)}\}$

$X^4 = S_1^2 \times\ S_2^2 = \{s_1^{2(1,\ 1)},\ s_1^{2(1,\ 2)},\ s_1^{2(2,\ 1)},\ s_1^{2(2,\ 2)}\} \times\ \{s_2^{2(1,\ 1)},\ s_2^{2(1,\ 2)},\ s_2^{2(2,\ 1)},\ s_2^{2(2,\ 2)}\}$

Apply the algorithm on the reduced normal form.

At initial node ω, since S is strict for both players, select player 1. Then $\Psi_1(s) = \{\{S_1^1\},\ \{S_1^2\}\}$. Thus player 1 has 2 choices to make at ω, labeled S_1^1 and S_1^2 according to the algorithm and leads to node ζ^1 and ζ^2 respectively where $T(\zeta^\tau) = S_1^\tau \times S_2$, $\tau = 1,\ 2$.

At either node, player 2 is the only one who can move. For each $S_1^\tau \times S_2$, $\tau = 1,\ 2$, the unique strict information set for 2 which contains $S_1^\tau \times S_2$ is S, and so $X^{\zeta^\tau} = S$. Now, $\Psi_2(s) = \{\{S_2^1\},\ \{S_2^2\}\}$. Thus at the node ζ^τ, player 2 chooses between S_2^1 and S_2^2, leading to two nodes $\zeta^{1\tau}$ and $\zeta^{2\tau}$ with $T(\zeta^{1\tau}) = S_1^\tau \times S_2^1$ and $T(\zeta^{2\tau}) = S_1^\tau \times S_2^2$, $\tau = 1,\ 2$.

Both players can move at $\zeta^{1\tau}$ and $\zeta^{2\tau}$ for $\tau = 1,\ 2$. Now $X^{\zeta^{11}} = X^1$, $X^{\zeta^{21}} = X^2$, $X^{\zeta^{12}} = X^3$ and $X^{\zeta^{22}} = X^4$. As $\Psi_1(X^1) = \{\{s_1^{1(1,1)},\ s_1^{1(1,2)}\},\ \{s_1^{1(2,\ 1)},\ s_1^{1(2,\ 2)}\}\}$, player 1 chooses between $\{s_1^{1(1,1)},\ s_1^{1(1,2)}\}$ and $\{s_1^{1(2,\ 1)},\ s_1^{1(2,\ 2)}\}$ at ζ^{11}. Similarly, since $\Psi_1(X^2) = \{\{s_1^{1(1,1)},\ s_1^{1(2,1)}\},\ \{s_1^{1(1,2)},\ s_1^{1(2,2)}\}\}$, player 1 chooses between $\{s_1^{1(1,1)},\ s_1^{1(2,1)}\}$ and $\{s_1^{1(1,2)},\ s_1^{1(2,2)}\}$ at ζ^{21}. As $\Psi_1(X^3) = \{\{s_1^{2(1,1)},\ s_1^{2(1,2)}\},\ \{s_1^{2(2,\ 1)},\ s_1^{2(2,\ 2)}\}\}$, player 1 chooses between $\{s_1^{2(1,1)},\ s_1^{2(1,2)}\}$ and $\{s_1^{2(2,\ 1)},\ s_1^{2(2,\ 2)}\}$ at ζ^{12}. And as $\Psi_1(X^4) = \{\{s_1^{2(1,1)},\ s_1^{2(2,1)}\},\ \{s_1^{2(1,2)},\ s_1^{2(2,2)}\}\}$, player 1 chooses between $\{s_1^{2(1,1)},\ s_1^{2(2,1)}\}$ and $\{s_1^{2(1,2)},\ s_1^{2(2,2)}\}$ at ζ^{22}. Denote the two nodes following $\zeta^{\kappa\tau}$ by $\zeta^{1\kappa\tau}$ and $\zeta^{2\kappa\tau}$ for $\kappa,\ \tau = 1,\ 2$.

Then,

$T(\zeta^{1,1,1}) = \{s_1^{1(1,\ 1)},\ s_1^{1(1,\ 2)}\} \times S_2^1$, $T(\zeta^{2,1,1}) = \{s_1^{1(2,\ 1)},\ s_1^{1(2,\ 2)}\} \times S_2^1$

$T(\zeta^{1,2,1}) = \{s_1^{1(1,\ 1)},\ s_1^{1(2,\ 1)}\} \times S_2^2$, $T(\zeta^{2,2,1}) = \{s_1^{1(1,\ 2)},\ s_1^{1(2,\ 2)}\} \times S_2^2$

$T(\zeta^{1,1,2}) = \{s_1^{2(1,\ 1)},\ s_1^{2(1,\ 2)}\} \times S_2^1$, $T(\zeta^{2,1,2}) = \{s_1^{2(2,\ 1)},\ s_1^{2(2,\ 2)}\} \times S_2^1$

$T(\zeta^{1,2,2}) = \{s_1^{2(1,\ 1)},\ s_1^{2(2,\ 1)}\} \times S_2^2$, $T(\zeta^{2,2,2}) = \{s_1^{2(1,\ 2)},\ s_1^{2(2,\ 2)}\} \times S_2^2$

Only player 2 can move at those nodes. Now $X^{\zeta^{1,1,1}} = X^{\zeta^{2,1,1}} = X^1$, $X^{\zeta^{1,2,1}} = X^{\zeta^{2,2,1}} = X^2$, $X^{\zeta^{1,1,2}} = X^{\zeta^{2,1,2}} = X^3$ and $X^{\zeta^{1,2,2}} = X^{\zeta^{2,2,2}} = X^4$.

Thus at $\zeta^{1,1,1}$ and $\zeta^{2,1,1}$, player 2 is choosing from $\{s_2^{1(1,\ 1)},\ s_2^{1(1,\ 2)}\}$ and $\{s_2^{1(2,\ 1)},\ s_2^{1(2,\ 2)}\}$. At the node node reached by $T(.) = \{s_1^{1(1,\ 1)},\ s_1^{1(1,\ 2)}\} \times \{s_2^{1(1,\ 1)},\ s_2^{1(1,\ 2)}\}$, payoff a_1^1 and b_1^1 is assigned for player 1 and 2 respectively. Similarly, payoffs $(a_2^1,\ b_2^1)$ is assigned for node reached by $T(.) = \{s_1^{1(1,\ 1)},\ s_1^{1(1,\ 2)}\} \times \{s_2^{1(2,\ 1)},\ s_2^{1(2,\ 2)}\}$, payoffs $(a_3^1,\ b_3^1)$ is assigned for node reached by $T(.) = \{s_1^{1(2,\ 1)},\ s_1^{1(2,\ 2)}\} \times \{s_2^{1(1,\ 1)},\ s_2^{1(1,\ 2)}\}$ and payoffs $(a_4^1,\ b_4^1)$ is assigned for nodes reached by $T(.) = \{s_1^{1(2,\ 1)},\ s_1^{1(2,\ 2)}\} \times \{s_2^{1(2,\ 1)},\ s_2^{1(2,\ 2)}\}$.

At $\zeta^{1,2,1}$ and $\zeta^{2,2,1}$, player 2 is choosing from $\{s_2^{2(1,\ 1)},\ s_2^{2(1,\ 2)}\}$ and $\{s_2^{2(2,\ 1)},\ s_2^{2(2,\ 2)}\}$. At the node reached by $T(.) = \{s_1^{1(1,\ 1)},\ s_1^{1(2,\ 1)}\} \times \{s_2^{2(1,\ 1)},\ s_2^{2(1,\ 2)}\}$, payoff a_1^2 and b_1^2 is assigned for player 1 and 2 respectively. Similarly, payoffs $(a_2^2,\ b_2^2)$ is assigned for node reached by $T(.) = \{s_1^{1(1,\ 1)},\ s_1^{1(2,\ 1)}\} \times \{s_2^{2(2,\ 1)},\ s_2^{2(2,\ 2)}\}$, payoffs $(a_3^2,\ b_3^2)$ is assigned for node reached by $T(.) = \{s_1^{1(1,\ 2)},\ s_1^{1(2,\ 2)}\} \times \{s_2^{2(1,\ 1)},\ s_2^{2(1,\ 2)}\}$ and payoffs $(a_4^2,\ b_4^2)$ is assigned for nodes reached by $T(.) = \{s_1^{1(1,\ 2)},\ s_1^{1(2,\ 2)}\} \times \{s_2^{2(1,\ 1)},\ s_2^{2(1,\ 2)}\}$.

At $\zeta^{1,1,2}$ and $\zeta^{2,1,2}$, player 2 is choosing from $\{s_2^{1(1,\ 1)},\ s_2^{1(2,\ 1)}\}$ and $\{s_2^{1(1,\ 2)},\ s_2^{1(2,\ 2)}\}$. At the node reached by $T(.) = \{s_1^{2(1,\ 1)},\ s_1^{2(1,\ 2)}\} \times \{s_2^{1(1,\ 1)},\ s_2^{1(2,\ 1)}\}$, payoff a_1^3 and b_1^3 is assigned for player 1 and 2 respectively. Similarly, payoffs $(a_2^3,\ b_2^3)$ is assigned for node reached by $T(.) = \{s_1^{2(1,\ 1)},\ s_1^{2(1,\ 2)}\} \times \{s_2^{1(1,\ 2)},\ s_2^{1(2,\ 2)}\}$, payoffs $(a_3^3,\ b_3^3)$ is assigned for node reached by $T(.) = \{s_1^{2(2,\ 1)},\ s_1^{2(2,\ 2)}\} \times \{s_2^{1(1,\ 1)},\ s_2^{1(2,\ 1)}\}$ and payoffs $(a_4^3,\ b_4^3)$ is assigned for nodes reached by $T(.) = \{s_1^{2(2,\ 1)},\ s_1^{2(2,\ 2)}\} \times \{s_2^{1(1,\ 2)},\ s_2^{1(2,\ 2)}\}$.

At $\zeta^{1,2,2}$ and $\zeta^{2,2,2}$, player 2 is choosing from $\{s_2^{2(1,\ 1)},\ s_2^{2(2,\ 1)}\}$ and $\{s_2^{2(1,\ 2)},\ s_2^{2(2,\ 2)}\}$. At the node reached by $T(.) = \{s_1^{2(1,\ 1)},\ s_1^{2(2,\ 1)}\} \times \{s_2^{2(1,\ 1)},\ s_2^{2(2,\ 1)}\}$, payoff a_1^4 and b_1^4 is assigned for player 1 and 2 respectively. Similarly, payoffs $(a_2^4,\ b_2^4)$ is assigned for node reached by $T(.) = \{s_1^{2(1,\ 1)},\ s_1^{2(2,\ 1)}\} \times \{s_2^{2(1,\ 2)},\ s_2^{2(2,\ 2)}\}$, payoffs $(a_3^4,\ b_3^4)$ is assigned for node reached by $T(.) = \{s_1^{2(1,\ 2)},\ s_1^{2(2,\ 2)}\} \times \{s_2^{2(1,\ 1)},\ s_2^{2(2,\ 1)}\}$ and payoffs $(a_4^4,\ b_4^4)$ is assigned for nodes reached by $T(.) = \{s_1^{2(1,\ 2)},\ s_1^{2(2,\ 2)}\} \times \{s_2^{2(1,\ 2)},\ s_2^{2(2,\ 2)}\}$.

It remains to construct information sets. Since node ζ^1 and ζ^2 belong to player 2 satisfy $X^{\zeta^1} = X^{\zeta^2} = S$ and have no predecessors, they are grouped together into one information set. Since $\zeta^{\kappa\tau}$ belongs to player 1 have every $X^{\zeta^{\kappa\tau}}$ assigned with different X^i, each of them consist one information set. For each $\zeta^{\rho\kappa\tau}$ belongs to player 2 with $X^{\zeta^{\rho\kappa\tau}}$ corresponding to X^i, they have the same predecessor $\zeta^{\kappa\tau}$ which corresponds to X^i. Thus they are grouped together as one information set. For example $\zeta^{1,1,1}$ and $\zeta^{2,1,1}$ are grouped together as one information set.

It is easy to see that the extensive form obtained from applying the algorithm on the two-stage PRNF yields the same tree structure as the original extensive form with difference only in the way actions are labeled.

Moreover, for general multistage game with distinct payoff assigned at the final nodes, the representation algorithm applied on the PRNF will recover the exact extensive form as the original one with difference in the way actions are labeled.

3.5 Example Revisited

We explicitly show here that the example exhibits restricted strategic complements. Figure 7 is the reduced normal form of the two-stage 2×2 game.

Single crossing condition is satisfied applying to $\{s_1^{1(1,2)},\ s_1^{2(1,\ 2)}\} \times \{s_2^{1(1,2)},\ s_2^{2(1,\ 2)}\}$ as $U_1(s_1^{1(1,2)},\ s_2^{1(1,2)}) = 15 > \ U_1(s_1^{2(1,2)},\ s_2^{1(1,2)}) = 2$ and $U_1(s_1^{1(1,2)},\ s_2^{2(1,2)}) = 13 > U_1(s_1^{2(1,2)},\ s_2^{2(1,2)}) = 4$. No crossing condition is satisfied as $11 = \ min\{15,\ 11\} > U_1(s_1,\ s_2),\ s_1 \in S_1^2$ and $s_2 \in S_2^1$ and $9 = \ min\{13,\ 9\} > \ U_1(s_1,\ s_2),\ s_1 \in S_1^2$ and $s_2 \in S_2^2$. Thus this two-stage game exhibits strategic complements.

Chapter 4

4 Multi-stage Games with Strategic Complements

In this paper, we are trying to study what additional structure strategic complementarity property implied on general multi-stage extensive form games. We will soon discover that for stage game to exhibit strategic complements, interesting common structure on best response sets arises. This will promotes a better understanding of dynamic games with strategic complements.

As noted in previous paper, even in the simple framework, the complicated notations require lots of attention. The need to deal with more general extensive form games calls for a better notation system. The following section proposed a way to index information sets in the stage game by identifying their corresponding strategies. By connecting information set with their corresponding set of strategies, we can facilitate the study of strategic complementarity which is a property essentially about the structure of best responses sets.

4.1 Multi-stage Game

At each stage t, a 2-player simultaneous game is played in which player 1 and 2 is choosing from action set A_i^t, $i = 1,\ 2$. In stage t, player i's action set is $A_i^t = \{a_i^{t,\ 1},\ a_i^{t,\ 2}, ...,\ a_i^{t,\ n^t}\}$. We assume that $a_i^{t,\ m} \prec a_i^{t,\ n}$ if $m < n$. There are in total T stage games played. Payoff are assigned with respect to outcomes in the end of T^{th} stage. The set of all information sets is denoted as H, in particular, player 1's set of information sets is H_1 and player 2's set is H_2. Actions available on each information set h is denoted as set $A(h)$. H_i^t denotes the set of those information set $h \in H_i$ such that $A(h) = A_i^t$. And denote $H^t = H_1^t \cup\ H_2^t$. H^{T+1} denotes the set of the ending nodes where payoffs are assigned .

For arbitrary information set $h \in H$, $S_1^\Gamma(h)$ denote the set of player 1's strategies that are consistent with reaching h. Thus for arbitrary player 1's strategy in $s_1 \in S_1^\Gamma(h)$, there exists player 2's strategy $s_2 \in S_2^\Gamma$ such that information set h is reached on the path of profile $(s_1,\ s_2)$. Similarly, $S_2^\Gamma(h)$ denote the set of player 2's strategies that are consistent with reaching h. $S^\Gamma(h) = S_1^\Gamma(h) \times S_2^\Gamma(h)$.

Now define some relative positions between information sets in the T-stage game.

(i). For arbitrary $h \in H^t$ and $h' \in H^{t'}$ if $t = t'$ and $S^\Gamma(h) = S^\Gamma(h')$, that is, $S_1^\Gamma(h) = S_1^\Gamma(h')$ and $S_2^\Gamma(h) = S_2^\Gamma(h')$, then h and h' **are reached at the same time**, denoted as $h \sim\ h'$.

(ii). For arbitrary $h \in H^t$ and $h' \in H^{t'}$, if $t > t'$ and $S^\Gamma(h) \subsetneqq S^\Gamma(h')$, that is, $S_1^\Gamma(h) \subsetneqq S_1^\Gamma(h')$ and $S_2^\Gamma(h) \subsetneqq S_2^\Gamma(h')$, then h **is reached after** h', denoted as $h \succ h'$.

(iii). For arbitrary $h,\ h' \in H$, if $S^\Gamma(h) \bigcap S^\Gamma(h') = \emptyset$, that is, either $S_1^\Gamma(h) \bigcap S_1^\Gamma(h') = \emptyset$ or $S_2^\Gamma(h) \bigcap S_2^\Gamma(h') = \emptyset$, then h **is off the path of** h'**.**

(iv). $h \in H^t$ is **reached right after** $h' \in H^{t'}$, if h is reached after h' and $t = t' + 1$.

The stage game structure ensures the following claims to be true.

1. For arbitrary player 1's information set $h_1 \in H_1$, there exists a unique information set reached at the same time as h_1. Such information set belongs to the other player.

2. Given $(h_1,\ h_2) \in H^t$ such that $S^\Gamma(h_1) = S^\Gamma(h_2)$, for arbitrary $a_1^{t,\ m_1} \in A_1(h_1)$

and $a_2^{t,\ m_2} \in A_2(h_2)$, there exists a unique pair of information sets $(\hat{h}_1,\ \hat{h}_2)$ reached right after $(h_1,\ h_2)$ such that for every $s_1 \in S_1^\Gamma(\hat{h}_1)$ and for every $s_2 \in S_2^\Gamma(\hat{h}_2)$, $s_1(h_1) = a_1^{t,\ m_1}$ and $s_2(h_2) = a_2^{t,\ m_2}$.

3. For every $\hat{h} \in H^{t+1}$ and $(a_1^{t,\ m_1}, a_2^{t,\ m_2}) \in A_1^t \times A_2^t$, there exists unique $(h_1,\ h_2) \in H_1^t \times H_2^t$ such that $S^\Gamma(h_1) = S^\Gamma(h_2)$ and for all $s_1 \in S_1^\Gamma(\hat{h})$, $s_1(h_1) = a_1^{t,\ m_1}$ and for all $s_2 \in S_2^\Gamma(\hat{h})$, $s_2(h_2) = a_2^{t,\ m_2}$.

4. Pick arbitrary information sets $\hat{h}_1,\ \tilde{h}_1 \in H_1^t$, $\hat{h}_1$ is off the path of $\tilde{h}_1$, that is, either $S_1^\Gamma(\hat{h}_1^t) \cap S_1^\Gamma(\tilde{h}_1^t) = \emptyset$ or $S_2^\Gamma(\hat{h}_1^t) \cap S_2^\Gamma(\tilde{h}_1^t) = \emptyset$ or both.

Lemma 12. *There exists a unique path connecting arbitrary information sets in the extensive form of the multi-stage game to the initial node.*

Proof. For arbitrary $\hat{h} \in H$, suppose $\hat{h} \in H^t$, there exists $(h_1^{t-1},\ h_2^{t-1}) \in H^{t-1}$ such that $\hat{h}$ is reached after $(h_1^{t-1},\ h_2^{t-1}) \in H^{t-1}$ with unique $(a_1^{t-1,\ m_1^{t-1}}, a_2^{t-1,\ m_2^{t-1}}) \in A_1^{t-1} \times A_2^{t-1}$ such that for all $s_1 \in S_1^\Gamma(\hat{h})$, $s_1(h_1^{t-1}) = a_1^{t-1,\ m_1^{t-1}}$ and for all $s_2 \in S_2^\Gamma(\hat{h})$, $s_2(h_2^{t-1}) = a_2^{t-1,\ m_2^{t-1}}$.

Continue identify $(h_1^{t-2},\ h_2^{t-2}) \in H_1^{t-2}$ such that $(h_1^{t-1},\ h_1^{t-1})$ is reached right after $(h_1^{t-2},\ h_2^{t-2})$ with unique $(a_1^{t-2,\ m_1^{t-2}}, a_2^{t-2,\ m_2^{t-2}}) \in A_1^{t-2} \times A_2^{t-2}$ such that for all $s_1 \in S_1^\Gamma(h_1^{t-1})$, $s_1(h_1^{t-2}) = a_1^{t-2,\ m_1^{t-2}}$ and for all $s_2 \in S_2^\Gamma(h_2^{t-1})$, $s_2(h_2^{t-2}) = a_2^{t-2,\ m_2^{t-2}}$.

Continue until $(h^0,\ h_2^0)$ is reached on the path with the corresponding $(a_1^{1,\ m_1^1},\ a_2^{1,\ m_2^1})$ identified. Thus there exists a path connecting $\hat{h}$ to h^0 as $(a_1^{1,\ m_1^1},\ a_2^{1,\ m_1^1})$ at $(h^0,\ h_2^0)$, $(a_1^{2,\ m_2^2},\ a_2^{2,\ m_2^2})$ at $(h_1^2,\ h_2^2)$,..., $(a_1^{t-1,\ m_1^{t-1}},\ a_2^{t-1,\ m_2^{t-1}})$ at the corresponding $(h_1^{t-1},\ h_2^{t-1})$ in which $\hat{h}$ is reached right after.

To show uniqueness, suppose there exists two different paths reaching $\hat{h}$ from h^0. In particular, information sets reached on the two paths are same until stage $\hat{t}$, $\hat{t} < t$ in which $\tilde{h} \in H^{\hat{t}}$ is reached on path 1 and $\tilde{h}' \in H^{\hat{t}}$ is reached on path 2. Thus $\tilde{h}$ is off the path of $\tilde{h}'$ and $S^{\Gamma}(\tilde{h}) \cap S^{\Gamma}(\tilde{h}') = \emptyset$. Since $\hat{h}$ is reached after $\tilde{h}$ and $\tilde{h}'$, $S^{\Gamma}(\hat{h}) \subseteq S^{\Gamma}(\tilde{h})$ and $S^{\Gamma}(\hat{h}) \subseteq S^{\Gamma}(\tilde{h}')$. Contradiction. Thus the path leading to arbitrary information set in the multi-stage game is unique. ∎

Thus we can properly index all the information sets in the multi-stage game by identifying the unique path leading to that particular information set. Denoting $\overline{H}^{m_2^1(m_1^1),\ldots,\ m_2^{\hat{t}-1}(m_1^{\hat{t}-1})}$, $(m_2^t,\ m_1^t \in \{1,\ldots,\ n^t\})$ as the information sets in $H^{\hat{t}}$ identified in the following way. Starting from choosing $(a_1^{1,\ m_1^1},\ a_2^{1,\ m_2^1})$ at $(h^0,\ h_2^0)$ and identify the corresponding $(h_1^2,\ h_2^2)$ as those information sets in H^2 such that for every $s_1 \in S_1^{\Gamma}(h_1^2)$ and for every $s_2 \in S_2^{\Gamma}(h_2^2)$, $s_1(h^0) = a_1^{1,\ m_1^1}$ and $s_2(h_2^0) = a_2^{1,\ m_2^1}$. Choosing $(a_1^{2,\ m_1^2},\ a_2^{2,\ m_2^2})$ at the corresponding $(h_1^2,\ h_2^2)$ and identify the corresponding $(h_1^3,\ h_2^3)$ in stage 3 ,..., choosing $(a_1^{\hat{t}-1,\ m_1^{\hat{t}-1}},\ a_2^{\hat{t}-1,\ m_2^{\hat{t}-1}})$ at the corresponding $(h_1^{\hat{t}-1},\ h_2^{\hat{t}-1})$ and identify the corresponding $(h_1^{\hat{t}},\ h_2^{\hat{t}})$ in stage $\hat{t}$, thus $\overline{H}^{m_2^1(m_1^1),\ldots,\ m_2^{\hat{t}-1}(m_1^{\hat{t}-1})} = \{h_1^{\hat{t}},\ h_2^{\hat{t}}\}$.

The $\overline{H}$ construction can be related to the familiar subgame concept as a way to index subgames. $\overline{H}^{m_2^1(m_1^1),\ldots,\ m_2^{\hat{t}-1}(m_1^{\hat{t}-1})} \cap H_1 = h_1$ can be considered as the initial node in that subgame. Denote Subgame $\overline{H}^{m_2^1(m_1^1),\ldots,\ m_2^{\hat{t}-1}(m_1^{\hat{t}-1})}$ as the subgame that start with initial node $\overline{H}^{m_2^1(m_1^1),\ldots,\ m_2^{\hat{t}-1}(m_1^{\hat{t}-1})}$. $S_1^{\Gamma}(h_1)$ can be considered as the set of player 1's strategy that allows subgame $\overline{H}^{m_2^1(m_1^1),\ldots,\ m_2^{\hat{t}-1}(m_1^{\hat{t}-1})}$ to be reached on the path. Similarly, $S_2^{\Gamma}(h_1)$ is considered as the set of player 2's strategy that allows subgame $\overline{H}^{m_2^1(m_1^1),\ldots,\ m_2^{\hat{t}-1}(m_1^{\hat{t}-1})}$ to be reached on the path. Thus for every $s_1 \in S_1^{\Gamma}(h_1)$, there exists $s_2 \in S_2^{\Gamma}(h_2)$ such that Subgame $\overline{H}^{m_2^1(m_1^1),\ldots,\ m_2^{\hat{t}-1}(m_1^{\hat{t}-1})}$ is reached of the path of profile $(s_1,\ s_2)$. For every information set reached after $\overline{H}^{m_2^1(m_1^1),\ldots,\ m_2^{\hat{t}-1}(m_1^{\hat{t}-1})}$, they denote subgames of Subgame $\overline{H}^{m_2^1(m_1^1),\ldots,\ m_2^{\hat{t}-1}(m_1^{\hat{t}-1})}$.

For convenience, if $m_1^t = infA_1^t = a_1^{t,\ 1}$, then $m_2^t(m_1^t)$ is denoted as $\underline{m}_2^t$ and if $m_1^t = supA_1^t = a_1^{t,\ n^t}$, then $m_2^t(m_1^t)$ is denoted as $\overline{m}_2^t$.

For arbitrary $\hat{t} \in \{1, .., \ T\}$ and $\hat{h} \in H^{\hat{t}}$, there exists a unique sequence of information sets starting from $(h^0,\ h_2^0)$ reaching $(h_1^2,\ h_2^2) \in H^2$,..., and $(h_1^{\hat{t}-1},\ h_2^{\hat{t}-1}) \in H^{\hat{t}-1}$ on the path with $(a_1^{t,\ m_1^t},\ a_2^{t,\ m_2^t})$ played at each $(h_1^t,\ h_2^t)$ at which $\hat{h}$ is reached right after $(h_1^{\hat{t}-1},\ h_2^{\hat{t}-1})$, thus $\hat{h} \in \overline{\hat{h}}^{m_2^{\hat{t}-1}(m_1^{\hat{t}-1})}$.

Since $S_1^\Gamma(\hat{h})$ is defined as the set of player 1's strategies that are consistent with reaching $\hat{h}$ and Lemma 1 indicates that there is an unique path leading to $\hat{h}$, $S_1^\Gamma(\hat{h})$, $S_1^\Gamma(\hat{h})$ can also be represented as $\{s_1 \in S_1^\Gamma |\ s_1(h^0) = a_1^{1,\ m_1^1},\ s_1(h_1^2) = a_1^{2,\ m_1^2}, ...,\ s_1(h_1^{\hat{t}-1}) = a_1^{\hat{t}-1,\ m_1^{\hat{t}-1}}\}$.

Similarly, $S_2^\Gamma(\hat{h}) = \{s_2 \in S_2^\Gamma |\ s_2(h_2^0) = a_2^{1,\ m_2^1},\ s_2(h_2^2) = a_2^{2,\ m_2^2}, ...,\ s_2(h_2^{\hat{t}-1}) = a_2^{\hat{t}-1,\ m_2^{\hat{t}-1}}\}$.

For $\tilde{s}_1 \in BR^1(s_2)$, $U_1(\tilde{s}_1,\ s_2) \geq U_1(s_1,\ s_2)$ for every $s_1 \in S_1$. Record the corresponding $\tilde{s}_1(h^0) = a_1^{1,\ m_1^1}$ and $s_2(h_2^0) = a_2^{1,\ m_2^1}$, $\tilde{s}_1(\overline{H}^{m_2^1(m_1^1)} \cap H_1) = a_1^{2,\ m_1^2}$ and $s_2(\overline{H}^{m_2^1(m_1^1)} \cap H_2) = a_2^{2,\ m_2^2}$,..., $\tilde{s}_1(\overline{H}^{m_2^1(m_1^1),...,\ m_2^{T-1}(m_1^{T-1})} \cap H_1) = a_1^{T,\ m_1^T}$ and $s_2(\overline{H}^{m_2^1(m_1^1),...,\ m_2^{T-1}(m_1^{T-1})} \cap H_2) = a_2^{T,\ m_2^T}$. It consists a path of profile $(\tilde{s}_1,\ s_2)$.

For every $\hat{s}_1 \in S_1^\Gamma(\overline{H}^{m_2^1(m_1^1),...,\ m_2^{T-1}(m_1^{T-1}),\ m_2^T(m_1^T)})$, that is, $\hat{s}_1 \in \{s_1 |\ s_1(h^0) = a_1^{1,\ m_1^1}, s_1(\overline{H}^{m_2^1(m_1^1)} \cap H_1) = a_1^{2,\ m_1^2} ... s_1(\overline{H}^{m_2^1(m_1^1)...\ m_2^{T-1}(m_1^{T-1})} \cap H_1) = a_1^{T,\ m_1^T}\}$, profile $(\hat{s}_1,\ s_2)$ will have the same path as $(\tilde{s}_1,\ s_2)$. As $U_1(\hat{s}_1,\ s_2) = U_1(\tilde{s}_1,\ s_2)$, $\hat{s}_1 \in BR^1(s_2)$. Thus $S_1^\Gamma(\overline{H}^{m_2^1(m_1^1),...,\ m_2^T(m_1^T)}) \subseteq BR^1(s_2)$.

$\underline{s}_2 = infS_2^\Gamma$, that is, $\forall\ t \in T,\ \forall\ h_2^t \in H_2^t,\ \underline{s}_2(h_2^t) = a_2^{t,\ 1}$,

$\overline{s}_2 = supS_2^\Gamma$, that is, $\forall\ t \in T,\ \forall\ h_2^t \in H_2^t,\ \overline{s}_2(h_2^t) = a_2^{t,\ n^t}$,

Let $a_1^{1,\ i}$ be $inf_{s_1 \in BR^1(\underline{s}_2)} s_1(h^0)$ and $a_1^{1,\ j}$ be $sup_{s_1 \in BR^1(\overline{s}_2)} s_1(h^0)$. Lemma 2 indicates that for every $s_2 \in S_2^\Gamma$ and $s_1 \in BR^1(s_2)$, $a_1^{1,\ i} \preceq s_1(h^0) \preceq a_1^{1,\ j}$.

Lemma 13. *For* $\hat{s}_2,\ \tilde{s}_2 \in S_2^{\Gamma}$,

(i). If $\hat{s}_2(h_2^0) \neq \tilde{s}_2(h_2^0)$, *then all those information sets reached on the best response path to* $\hat{s}_2$ *and reached after* h^0 *are off the best response path to* $\tilde{s}_2$.

(ii). If $\hat{s}_2(h_2^0) = \tilde{s}_2(h_2^0) = a_1^{1,\ m_2^1}$ *and for all* $h_2 \in \bigcup_{m_1^1 \in \{1,\dots,\ n^1\}} \{h_2 \in H_2 |\ h_2 \succeq \overline{H}^{m_2^1(m_1^1)}\}$, $\hat{s}_2(h_2) \preceq \tilde{s}_2(h_2)$, *then* $BR^1(\hat{s}_2) \sqsubseteq BR^1(\tilde{s}_2)$.

Proof. (i) Pick arbitrary $\tilde{s}_1 \in BR^1(\tilde{s}_2)$ and $\hat{s}_1 \in BR^1(\hat{s}_2)$. Information sets reached right after h^0 and on the path of profile $(\tilde{s}_1,\ \tilde{s}_2)$ as $\tilde{h}$ and information sets reached on the path of profile $(\hat{s}_1,\ \hat{s}_2)$ as $\hat{h}$. $\tilde{h},\ \hat{h} \in H^2$. As $S_2^{\Gamma}(\tilde{h}) = \{s_2 \in S_2^{\Gamma} |\ s_2(h_2^0) = \tilde{s}_2(h_2^0)\}$, $S_2^{\Gamma}(\hat{h}) = \{s_2 \in S_2^{\Gamma} |\ s_2(h_2^0) = \hat{s}_2(h_2^0)\}$ and $\{s_2 \in S_2^{\Gamma} |\ s_2(h_2^0) = \tilde{s}_2(h_2^0)\} \cap \{s_2 \in S_2^{\Gamma} |\ s_2(h_2^0) = \hat{s}_2(h_2^0)\} = \emptyset$, it is easy to prove that $\tilde{h} \cap \hat{h} = \emptyset$. For all $\hat{h}' \in \{h |\ h \succ \hat{h}\}$, $S_2^{\Gamma}(\hat{h}') \subseteq S_2^{\Gamma}(\hat{h})$ and for all $\tilde{h}' \in \{h |\ h \succ \tilde{h}\}$, $S_2^{\Gamma}(\tilde{h}') \subseteq S_2^{\Gamma}(\tilde{h})$, thus $\hat{h}' \cap \tilde{h}' = \emptyset$ and $\{h |\ h \succeq \tilde{h}\} \cap \{h |\ h \succeq \hat{h}\} = \emptyset$. Thus after h^0 and h_2^0 all the information sets reached on the best response path to $\hat{s}_2$ are off the best response path to $\tilde{s}_2$.

(ii). Form $\hat{s}_2'$ such that for all $h_2 \in \bigcup_{m_1^1 \in \{1,\dots,\ n^1\}} \{h_2 \in H_2 |\ h_2 \succeq \overline{H}^{m_2^1(m_1^1)}\}$, $\hat{s}_2'(h_2) = \hat{s}_2(h_2)$ and for all $h_2 \in H_2 \backslash \bigcup_{m_1^1 \in \{1,\dots,\ n^1\}} \{h_2 \in H_2 |\ h_2 \succeq \overline{H}^{m_2^1(m_1^1)}\}$, $\hat{s}_2'(h_2) = infA(h_2)$. Since information sets in $H_2 \backslash \bigcup_{m_1^1 \in \{1,\dots,\ n^1\}} \{h_2 \in H_2 |\ h_2 \succeq \overline{H}^{m_2^1(m_1^1)}\}$ are off the paths of $(s_1,\ \hat{s}_2)$ for arbitrary $s_1 \in S_1^{\Gamma}$, $BR^1(\hat{s}_2) = BR^1(\hat{s}_2')$. Similarly, form $\tilde{s}_2'$ such that for all $h_2 \in \bigcup_{m_1^1 \in \{1,\dots,\ n^1\}} \{h_2 \in H_2 |\ h_2 \succeq \overline{H}^{m_2^1(m_1^1)}\}$, $\tilde{s}_2'(h_2) = \tilde{s}_2(h_2)$ and for all $h_2 \in H_2 \backslash \bigcup_{m_1^1 \in \{1,\dots,\ n^1\}} \{h_2 \in H_2 |\ h_2 \succeq \overline{H}^{m_2^1(m_1^1)}\}$, $\tilde{s}_2'(h_2) = supA(h_2)$. $BR^1(\tilde{s}_2) = BR^1(\tilde{s}_2')$. Since $\hat{s}_2' \preceq \tilde{s}_2'$, $BR^1(\hat{s}_2') \sqsubseteq BR^1(\tilde{s}_2')$. Thus $BR^1(\hat{s}_2) \sqsubseteq BR^1(\tilde{s}_2)$. ∎

4.2 Main Results

From now on, we will be looking at the best response structures with respect to opponents' strategies under strategic complementarity assumption.

Player 2's strategies are divided into three groups. Group 1 consist of extreme strategies $\underline{s}_2$ and $\overline{s}_2$. Group 2 consists of strategies that assigns different actions from both $\underline{s}_2$ and $\overline{s}_2$ at h_2^0. Group 3 consists of strategies that assigns the same actions as $\underline{s}_2$ or $\overline{s}_2$ at h_2^0. Common structures of best response sets within each groups will arise after strategic complementarities assumption is applied.

Denote $inf_{s_1 \in BR^1(\underline{s}_2)} s_1(h^0) = a_1^{1,\ i}$ and $sup_{s_1 \in BR^1(\overline{s}_2)} s_1(h^0) = a_1^{1,\ j}$,

Theorem 4. $BR^1(\underline{s}_2) = S_1^\Gamma(\overline{H}^{1(i),\ \underline{1}^2,\dots,\ \underline{1}^T}) \cup A_1 \cup B$ *with* $A_1 \subseteq \{S_1^\Gamma(\overline{H}^{1(i),\underline{1}^2,\dots,\ \underline{1}^{\hat{t}-1},\ 1(m^{\hat{t}})})|\ \hat{t} \in \{2,\dots,\ T\},\ a_1^{1,\ m_1^{\hat{t}}} \in A_1^{\hat{t}}\}$ *and* $B \subseteq \{S_1^\Gamma(\overline{H}^{1(m_1^1)})|\ a_1^{1,\ i} \prec a_1^{1,\ m_1^1} \preceq a_1^{1,\ j}\}$.

Proof. To prove the theorem, we want to prove the following claims:
(i). $S_1^\Gamma(\overline{H}^{1(i),\ \underline{1}^2,\dots,\ \underline{1}^T}) \subseteq BR^1(\underline{s}_2)$
(ii). If there exists $\hat{s}_1 \in BR^1(\underline{s}_2)$, $\hat{s}_1(h^0) \succ a_1^{1,\ i}$, then $\{s_1 \in S_1^\Gamma|\ s_1(h^0) = \hat{s}_1(h^0)\} \subseteq BR^1(\underline{s}_2)$.
(iii). If there exists $s_1 \in BR^1(\underline{s}_2)$ such that $s_1 \in S_1^\Gamma(\overline{H}^{1(i),\ \underline{1}^2,\dots,\ \underline{1}^{\hat{t}-1}})$ and $s_1(\overline{H}^{1(i),\ \underline{1}^2,\dots,\ \underline{1}^{\hat{t}-1}} \cap H_1) = a_1^{\hat{t},\ m_1^{\hat{t}}} \in A_1^{\hat{t}} \backslash a_1^{\hat{t},\ 1}$, then $S_1^\Gamma(\overline{H}^{1(i),\ \underline{1}^2,\dots,\ \underline{1}^{\hat{t}-1},\ 1(m_1^{\hat{t}})}) \subset BR^1(\underline{s}_2)$.

(i). Pick arbitrary $\tilde{s}_1 \in BR^1(\overline{s}_2)$ and $\hat{s}_1 \in BR^1(\underline{s}_2)$ such that $\tilde{s}_1(h^0) = a_1^{1,\ j}$ and $\hat{s}_1(h^0) = a_1^{1,\ i}$. Information set $\overline{H}^{n^1(j)}$ is reached on the path of profile $(\tilde{s}_1,\ \overline{s}_2)$ and Information set $\overline{H}^{1(i)}$ is reached on the path of profile $(\hat{s}_1,\ \underline{s}_2)$. Lemma 2 indicates that information sets $\overline{H}^{1(i)}$ are off all the best response paths with respect to $\overline{s}_2$. Thus in response to $\overline{s}_2$, player 1 is indifferent among the choices at $\overline{H}^{1(i)} \cap H_1$.

Pick arbitrary $\tilde{s}_1 \in BR^1(\overline{s}_2)$ and $\hat{s}_1 \in BR^1(\underline{s}_2)$ and let $\tilde{s}_1(\overline{H}^{1(i)} \cap H_1) = a_1^{2,\ 1}$. Strategic complementarities imply that $\tilde{s}_1 \wedge \hat{s}_1 \in BR^1(\underline{s}_2)$, in particular, $\tilde{s}_1 \wedge \hat{s}_1(h^0) = \tilde{s}_1(h^0) \wedge \hat{s}_1(h^0) = a_1^{1,\ i}$, $\tilde{s}_1 \wedge \hat{s}_1(\overline{H}^{1(i)} \cap H_1) = a_1^{2,\ 1}$. Thus there exists $s_1 \in BR^1(\underline{s}_2)$ such that $s_1(h^0) = a_1^{1,\ i}$ and $s_1(\overline{H}^{1(i)} \cap H_1) = a_1^{2,\ 1}$.

Suppose there exists $\hat{s}_1 \in BR^1(\underline{s}_2)$, $\hat{s}_1(h^0) = a_1^{1,\ i}$, $\hat{s}_1(\overline{H}^{1(i)} \cap H_1) = a_1^{2,\ 1}$, $\hat{s}_1(\overline{H}^{1(i),\ \underline{1}^2} \cap H_1) = a_1^{3,\ 1}$ ... $\hat{s}_1(\overline{H}^{1(i),...,\ \underline{1}^{t-2}} \cap H_1) = a_1^{t-1,\ 1}$. As $\overline{H}^{1(i),\underline{1}^2,...,\ \underline{1}^{t-1}} \succ \overline{H}^{1(i)}$ and $\overline{H}^{1(i)}$ are off all the best response paths with respect to $\overline{s}_2$, player 1 is indifferent among the choices on $\overline{H}^{1(i),\underline{1}^2,...,\ \underline{1}^{t-1}} \cap H_1$. Pick arbitrary $\tilde{s}_1 \in BR^1(\overline{s}_2)$, $\tilde{s}_1(\overline{H}^{1(i),...,\ \underline{1}^{t-1}} \cap H_1) = a_1^{t,\ 1}$.

Strategic complementarities imply that $\tilde{s}_1 \wedge \hat{s}_1 \in BR^1(\underline{s}_2)$, in particular, $\tilde{s}_1 \wedge \hat{s}_1(\overline{H}^{1(i),...,\ \underline{1}^{t-1}} \cap H_1) = a_1^{t,\ 1}$. Thus there exists $s_1 \in BR^1(\underline{s}_2)$, $s_1(h^0) = a_1^{1,\ i}$, $s_1(\overline{H}^{1(i)} \cap H_1) = a_1^{2,\ 1}$... $s_1(\overline{H}^{1(i),...,\ \underline{1}^{t-1}} \cap H_1) = a_1^{t,\ 1}$.

Let $t = T$, there exists $s_1 \in BR^1(\underline{s}_2)$, $s_1(h^0) = a_1^{1,\ i}$, $s_1(\overline{H}^{1(i)} \cap H_1) = a_1^{2,\ 1}$... $s_1(\overline{H}^{1(i),...,\ \underline{1}^{T-1}} \cap H_1) = a_1^{T,\ 1}$. This is a complete path of profile $(s_1,\ \underline{s}_2)$. Thus those player 1's strategy that together with $\underline{s}_2$ have the same path as $(s_1,\ \underline{s}_2)$ are also player 1's best responses to $\underline{s}_2$. Thus $\{s_1 \in S_1^\Gamma |\ s_1(h^0) = a_1^{1,\ i},\ s_1(\overline{H}^{1(i)} \cap H_1) = a_1^{2,\ 1}... \ s_1(\overline{H}^{1(i),...,\ \underline{1}^{T-1}} \cap H_1) = a_1^{T,\ 1}\} \subseteq BR^1(\underline{s}_2)$. In particular, $S_1^\Gamma(\overline{H}^{1(i),...,\ \underline{1}^{T-1},\ \underline{1}^T}) \subseteq BR^1(\underline{s}_2)$

(ii). Form $\tilde{s}_2$ such that $\tilde{s}_2(h_2^0) = a_2^{1,\ 1}$ and for all $m \in \{1, ...,\ n^1\}$ and $h_2 \succeq \overline{H}^{1(m)} \cap H_2$, $\tilde{s}_2(h_2) = \underline{s}_2(h_2)$ and for all $n \in \{2, ...,\ n^1\}$ and $h_2 \succeq \overline{H}^{n(m)} \cap H_2$, $\tilde{s}_2(h_2) = \overline{s}_2(h_2)$. Since information sets $\overline{H}^{n(m)}$ are off the path of profile $(s_1,\ \tilde{s}_2)$ for arbitrary $s_1 \in S_1^\Gamma$, $BR^1(\underline{s}_2) = BR^1(\tilde{s}_2)$. $\underline{s}_2 \prec \tilde{s}_2$ and strategic complementarities imply that $BR^1(\underline{s}_2)$ is a lattice.

Since $\hat{s}_1 \in BR^1(\underline{s}_2)$ and $\hat{s}_1(h^0) \succ a_1^{1,\ i}$, denote $\hat{s}_1(h^0) = a_1^{1,\ t}$ with $t > i$. Lemma 4 implies that $S_1^{\Gamma}(\overline{H}^{1(i),\ldots,\ \underline{1}^{T-1},\ \underline{1}^{T}}) \subseteq BR^1(\underline{s}_2)$ and $S_1^{\Gamma}(\overline{H}^{1(t),\ldots,\ \underline{1}^{T-1},\ \underline{1}^{T}}) \subseteq BR^1(\underline{s}_2)$. Pick arbitrary $\tilde{s}_1 \in S_1^{\Gamma}(\overline{H}^{1(i),\ldots,\ \underline{1}^{T-1},\ \underline{1}^{T}})$. Since $\overline{H}^{1(t)}$ are off the path of profile $(\tilde{s}_1,\ \underline{s}_2)$, all $h_1 \succeq \overline{H}^{1(t)} \cap H_1$ are off the path of that profile, thus assign $\tilde{s}_1(h_1)$ with arbitrary actions in $A(h_1)$. Pick $\tilde{s}_1' = inf S_1^{\Gamma}(\overline{H}^{1(t),\ldots,\ \underline{1}^{T-1},\ \underline{1}^{T}})$. Thus $\tilde{s}_1 \vee \tilde{s}_1' \in BR^1(\underline{s}_2)$. In particular, $(\tilde{s}_1 \vee \tilde{s}_1')(h^0) = a_1^{1,\ t}$ thus $\overline{H}^{1(t)}$ is reached on the path of $(\tilde{s}_1 \vee \tilde{s}_1',\ \underline{s}_2)$. And for all $h_1 \succeq \overline{H}^{1(t)}$, $(\tilde{s}_1 \vee \tilde{s}_1')(h_1) = \tilde{s}_1(h_1)$. Since $\tilde{s}_1$ assigns arbitrary action at the information sets reached after $\overline{H}^{1(t)}$, $S_1^{\Gamma}(\overline{H}^{1(t)}) \subseteq BR^1(\underline{s}_2)$. Thus (ii) is proved.

(iii). Pick such s_1 in the assumption, then $s_1 \in S_1^{\Gamma}(\overline{H}^{1(i),\ldots,\ \underline{1}^{\hat{t}-1},\ 1(m_1^{\hat{t}})})$. Pick arbitrary $\tilde{s}_1 \in S_1^{\Gamma}(\overline{H}^{1(i),\ldots,\ \underline{1}^{T-1},\ \underline{1}^{T}})$ such that $\tilde{s}_1(\overline{H}^{1(i),\ldots,\ \underline{1}^{\hat{t}-1},\ 1(m_1^{\hat{t}})} \cap H_1) = a_1^{\hat{t}+1,\ n^{\hat{t}+1}}$, it is a reasonable assumption as information set $\overline{H}^{1(i),\ldots,\ \underline{1}^{\hat{t}-1},\ 1(m_1^{\hat{t}})}$ is off the path of profile $(\tilde{s}_1,\ \underline{s}_2)$. Thus (i) implies $\tilde{s}_1 \in BR^1(\underline{s}_2)$. Thus $\tilde{s}_1 \vee s_1 \in BR^1(\underline{s}_2)$, in particular, $\tilde{s}_1 \vee s_1 \in S_1^{\Gamma}(\overline{H}^{1(i),\ldots,\ \underline{1}^{\hat{t}-1},\ 1(m_1^{\hat{t}})})$ and $\tilde{s}_1 \vee s_1(\overline{H}^{1(i),\ldots,\ \underline{1}^{\hat{t}-1},\ 1(m_1^{\hat{t}})} \cap H_1) = a_1^{\hat{t}+1,\ n^{\hat{t}+1}}$.

Pick arbitrary $\hat{s}_1 \in BR^1(\overline{s}_2)$, let $\hat{s}_1(\overline{H}^{1(i),\ldots,\ \underline{1}^{\hat{t}-1}} \cap H_1) = a_1^{\hat{t},\ m_1^{\hat{t}}}$ and $\hat{s}_1(\overline{H}^{1(i),\ldots,\ \underline{1}^{\hat{t}-1},\ 1(m_1^{\hat{t}})} \cap H_1) = a_1^{\hat{t}+1,\ m_1^{\hat{t}+1}}$. Strategic complementarities and $\underline{s}_2 \prec \overline{s}_2$ implies that $(\tilde{s}_1 \vee s_1) \wedge \hat{s}_1 \in BR^1(\underline{s}_2)$.

Thus for arbitrary $a_1^{\hat{t}+1,\ m_1^{\hat{t}+1}} \in A_1^{\hat{t}+1}$, there exists $s_1 \in BR^1(\underline{s}_2)$ such that $s_1(\overline{H}^{1(i),\ldots,\ \underline{1}^{\hat{t}-1},\ 1(m_1^{\hat{t}})} \cap H_1) = a_1^{\hat{t}+1, m_1^{\hat{t}+1}}$.

Suppose for arbitrary $a_1^{\hat{t}+1,\ m_1^{\hat{t}+1}} \in A_1^{\hat{t}+1}, \ldots, a_1^{t-1,\ m_1^{t-1}} \in A_1^{t-1}$, $t-1 > \hat{t}+1$ there exists $s_1 \in BR^1(\underline{s}_2)$ such that $s_1 \in S_1^{\Gamma}(\overline{H}^{1(i),\ldots,\ \underline{1}^{\hat{t}-1},\ 1(m_1^{\hat{t}}),\ 1(m_1^{\hat{t}+1}),\ldots,\ 1(m_1^{t-1})})$.

Let $\tilde{s}_1 \in BR^1(\underline{s}_2)$ such that $\tilde{s}_1 \in S_1^{\Gamma}(\overline{H}^{1(i),\ldots,\ \underline{1}^{T-1},\ \underline{1}^{T}})$. As information sets $\overline{H}^{1(i),\ldots,\ \underline{1}^{\hat{t}-1},\ 1(m_1^{\hat{t}})}$ are off the path of profile $(\tilde{s}_1,\ \underline{s}_2)$, player 1 is indifferent among the choices on those information sets. Let $\tilde{s}_1(\overline{H}^{1(i),\ldots,\ \underline{1}^{\hat{t}-1},\ 1(m_1^{\hat{t}})} \cap H_1) = s_1(\overline{H}^{1(i),\ldots,\ \underline{1}^{\hat{t}-1},\ 1(m_1^{\hat{t}})} \cap H_1), \ldots, \tilde{s}_1(\overline{H}^{1(i),\ldots,\ \underline{1}^{\hat{t}-1},\ 1(m_1^{\hat{t}}),\ 1(m_1^{\hat{t}+1}),\ldots,\ 1}$

$H_1) = s_1(\overline{H}^{1(i),..., \underline{1}^{\hat{t}-1},\ 1(m_1^{\hat{t}}),\ 1(m_1^{\hat{t}+1}),.., 1(m_1^{t-2})} \cap H_1)$ and $\tilde{s}_1(\overline{H}^{1(i),..., \underline{1}^{\hat{t}-1},\ 1(m_1^{\hat{t}}),\ 1(m_1^{\hat{t}+1}),.., 1(m_1^{t-1})} \cap H_1) = a_1^{t,\ n^t}$. Strategic complementarities implies that $\tilde{s}_1 \vee s_1 \in BR^1(s_2)$. Thus $(\tilde{s}_1 \vee \hat{s}_1) \in S_1^{\Gamma}(\overline{H}^{1(i),..., \underline{1}^{\hat{t}-1},\ 1(m_1^{\hat{t}}),\ 1(m_1^{\hat{t}+1}),.., 1(m_1^{t-1})})$ and $(\tilde{s}_1 \vee \hat{s}_1)(\overline{H}^{1(i),..., \underline{1}^{\hat{t}-1},\ 1(m_1^{\hat{t}}),\ 1(m_1^{\hat{t}+1}),.., 1(m_1^{t-1})} \cap H_1) = a_1^{t,\ n^t}$.

Let $\hat{s}_1 \in BR^1(\overline{s}_2)$. As information sets $\overline{H}^{1(i),..., \underline{1}^{\hat{t}-1},\ 1(m_1^{\hat{t}})}$ is off the path of profile $(\hat{s}_1,\ \overline{s}_2)$, player 1 is indifferent among the choices on those information sets.

Let $\hat{s}_1(\overline{H}^{1(i),..., \underline{1}^{\hat{t}-1},\ 1(m_1^{\hat{t}})} \cap H_1) = s_1(\overline{H}^{1(i),..., \underline{1}^{\hat{t}-1},\ 1(m_1^{\hat{t}})} \cap H_1)$,...,$\hat{s}_1(\overline{H}^{1(i),..., 1(m_1^{\hat{t}}),\ 1(m_1^{\hat{t}+1}),.., 1(m_1^{t-2})} \cap H_1) = s_1(\overline{H}^{1(i),..., 1(m_1^{\hat{t}}),\ 1(m_1^{\hat{t}+1}),.., 1(m_1^{t-2})} \cap H_1)$ and $\hat{s}_1(\overline{H}^{1(i),..., \underline{1}^{\hat{t}-1},\ 1(m_1^{\hat{t}}),\ 1(m_1^{\hat{t}+1}),.., 1(m_1^{t-1})} \cap H_1) = a_1^{t,\ m_1^t}$, $m_1^t \in A_1^t$. $\overline{s}_2 \succ s_2$ and strategic complementarities implies that $(\tilde{s}_1 \vee s_1) \wedge \hat{s}_1 \in BR^1(s_2)$. Thus $(\tilde{s}_1 \vee s_1) \wedge \hat{s}_1 \in S_1^{\Gamma}(\overline{H}^{1(i),..., 1(m_1^{\hat{t}}),\ 1(m_1^{\hat{t}+1}),.., 1(m_1^{t-1})})$ and $(\tilde{s}_1 \vee s_1) \wedge \hat{s}_1(\overline{H}^{1(i),..., 1(m_1^{\hat{t}}),\ 1(m_1^{\hat{t}+1}),.., 1(m_1^{t-1})} \cap H_1) = a_1^{t,\ m_1^t}$ for arbitrary $m_1^t \in \{1,...,\ n^t - 1\}$.

Thus for arbitrary $a_1^{\hat{t}+1,\ m_1^{\hat{t}+1}} \in A_1^{\hat{t}+1}$,..., $a_1^{t,\ m_1^t} \in A_1^t$, there exists $s_1 \in BR^1(\underline{s}_2)$ such that $s_1 \in S_1^{\Gamma}(\overline{H}^{1(i),..., \underline{1}^{\hat{t}-1},\ 1(m_1^{\hat{t}}),\ 1(m_1^{\hat{t}+1}),.., 1(m_1^{t})})$.

At $t = T$, for arbitrary $m_1^{\hat{t}+1} \in \{1,...,\ n^{\hat{t}+1}\}$,..., $m_1^T \in \{1,...,\ n^T\}$, there exists $s_1 \in BR^1(\underline{s}_2)$ such that $\hat{s}_1(h^0) = a_1^{1,\ i}$, $\hat{s}_1(\overline{H}^{1(i)} \cap H_1) = a_1^{2,\ 1}$,..., $\hat{s}_1(\overline{H}^{1(i),..., m_2^{T-1}(m_1^{T-1})} \cap H_1) = a_1^{T,\ m_1^T}$. As this denote one complete path of profile of $(s_1,\ \underline{s}_2)$, it is easy to see that $\{s_1 \in S_1^{\Gamma} |\ s_1(h^0) = a_1^{1,\ i},\ s_1(\overline{H}^{1(i)} \cap H_1) = a_1^{2,\ 1}, ...,\ s_1(\overline{H}^{m_2^1(m_1^1),..., m_2^{T-1}(m_1^{T-1})} \cap H_1) = a_1^{T,\ m_1^T}\} \subseteq BR^1(\underline{s}_2)$.

Thus $S_1^{\Gamma}(\overline{H}^{1(i),..., \underline{1}^{\hat{t}-1},\ 1(m_1^{\hat{t}})}) = \bigcup_{m_1^{\hat{t}+1}} \cdots \bigcup_{m_1^T} \{s_1 \in S_1^{\Gamma}(\overline{H}^{1(i),..., \underline{1}^{\hat{t}-1},\ 1(m_1^{\hat{t}})}) | s_1(\overline{H}^{1(i),..., 1(m_1^{\hat{t}})}) = a_1^{\hat{t}+1,\ m_1^{\hat{t}+1}},\ ...,\ s_1(\overline{H}^{1(i),..., \underline{1}^{\hat{t}-1},\ 1(m_1^{\hat{t}}),\ 1(m_1^{\hat{t}+1}),.., 1(m_1^{T-1})} \cap H_1) = a_1^{T,\ m_1^T}\} \subseteq BR^1(s_2)$. ∎

Corollary 7. *Consider the following,*

(i). For every $s_2 \in S_2^{\Gamma}(\overline{H}^{1(i),\ \underline{1}^2,..., \underline{1}^T})$*, either* $BR^1(\underline{s}_2) \subseteq BR^1(s_2)$ *or* $BR^1(s_2) \cap \{s_1 \in$

$S_1^\Gamma|\ s_1(h^0) = a^{1,\ i}\} = \emptyset$.

(ii). For every $s_2 \in S_2^\Gamma(\overline{H}^{n^1(j),\ \overline{n}^2,...,\ \overline{n}^T})$, *either* $BR^1(\overline{s}_2) \subseteq BR^1(s_2)$ *or* $BR^1(s_2) \cap \{s_1 \in S_1^\Gamma|\ s_1(h^0) = a^{1,\ j}\} = \emptyset$.

Proof. (i). For $s_2 \in S_2^\Gamma(\overline{H}^{1(i),\ \underline{1}^2,...,\ \underline{1}^T})$, if $BR^1(s_2) \cap \{s_1 \in S_1^\Gamma|\ s_1(h^0) = a_1^{1,\ i}\} \neq \emptyset$, then $S_1^\Gamma(\overline{H}^{1(i),\ \underline{1}^2,...,\ \underline{1}^T}) \subseteq BR^1(s_2)$. Pick arbitrary $\hat{s}_1 \in BR^1(\underline{s}_2)$ and $\tilde{s}_1 = inf S_1^\Gamma(\overline{H}^{1(i),\ \underline{1}^2,...,\ \underline{1}^T})$. $s_2 \succeq \underline{s}_2$ and strategic complementarities imply that $\hat{s}_1 \vee \tilde{s}_1 = \hat{s}_1 \in BR^1(s_2)$. Thus $BR^1(\underline{s}_1) \subseteq BR^1(s_2)$.

(ii). Similar to (i). ∎

Corollary 8. $BR^1(\overline{s}_2) = S_1^\Gamma(\overline{H}^{n^1(j),\ \overline{n}^2,...,\ \overline{n}^T}) \cup A_1 \cup B$ *with* $A_1 \subseteq \{S_1^\Gamma(\overline{H}^{n^1(j),\ \overline{n}^2,...,\ \overline{n}^{\hat{t}-1},\ n^{\hat{t}}(m_1^{\hat{t}})})|\ \hat{t} \in \{2, ..., T\},\ a_1^{\hat{t},\ m_1^{\hat{t}}} \in A_1^{\hat{t}}\}$ *and* $B \subseteq \{S_1^\Gamma(\overline{H}^{n^1(m_1^1)})|\ a_1^{1,\ i} \preceq a_1^{1,\ m_1^1} \prec a_1^{1,\ j}\}$

Proof. Proved similarly as Theorem 1. ∎

Theorem 1 implies that if Subgame $\overline{H}^{1(i)}$ is reached on some best response paths of $\underline{s}_2$, then Subgame $\overline{H}^{1(i),...,\ \underline{1}^{T-1}}$ can be reached on some best response paths of $\underline{s}_2$. Similarly, if Subgame $\overline{H}^{n^1(j)}$ is reached on some best response paths of $\overline{s}_2$, then Subgame $\overline{H}^{n^1(j),...,\ \overline{n}^{T-1}}$ can be reached on some best response paths of $\overline{s}_2$.

The following denotes an important structure of $S_1^\Gamma(h)$ that will be used in later proofs.

For a given $\hat{s}_2$ and $\hat{h} = (\hat{h}_1,\ \hat{h}_2) \in H^{\hat{t}}$ such that $\hat{s}_2 \in S_2^\Gamma(\hat{h})$, $S_1^\Gamma(\hat{h}) = \bigcup_{m_1^{\hat{t}} \in \{1...\ n^{\hat{t}}\}} \{s_1 \in S_1^\Gamma(\hat{h})|\ s_1(\hat{h}_1) = a_1^{\hat{t},\ m_1^{\hat{t}}}\}$. Since $\hat{s}_2(\hat{h}_2) = a_2^{\hat{t},\ m_2^{\hat{t}}}$, for each $a_1^{\hat{t},\ m_1^{\hat{t}}} \in A_1^{\hat{t}}$, player 1's information set reached right after $\hat{h}$ is $h_1^{\hat{t}+1} = \overline{\hat{h}}^{m_2^{\hat{t}}(m_1^{\hat{t}})} \cap H_1$. Thus $S_1^\Gamma(\hat{h}) = \bigcup_{m_1^{\hat{t}} \in \{1...\ n^{\hat{t}}\}} \bigcup_{m_1^{\hat{t}+1} \in \{1...\ n^{\hat{t}+1}\}} \{s_1 \in S_1^\Gamma(\hat{h})|\ s_1(\hat{h}_1) = a_1^{\hat{t},\ m_1^{\hat{t}}},\ s_1(\overline{\hat{h}}^{m_2^{\hat{t}}(m_1^{\hat{t}})} \cap H_1) = a_1^{\hat{t}+1,\ m_1^{\hat{t}+1}}\}$.

In general, $S_1^\Gamma(\hat{h}) = \bigcup_{m_1^{\hat{t}} \in \{1...\ n^{\hat{t}}\}} \cdots \bigcup_{m_1^t \in \{1...\ n^t\}} \{s_1 \in S_1^\Gamma(\hat{h})|\ s_1(\hat{h}_1) = a_1^{\hat{t},\ m_1^{\hat{t}}} ...\ s_1(h_1^t) =$

$a_1^{t,\ m_1^t}\}$, with $\hat{s}_2(\hat{h}_2) = a_2^{\hat{t},\ m_2^{\hat{t}}} \in A_2^{\hat{t}}$, $h^{\hat{t}+1} = \overline{\hat{h}}^{m_2^{\hat{t}}(m_1^{\hat{t}})}$, $\hat{s}_2(h_2^{\hat{t}+1}) = a_2^{\hat{t}+1,\ m_2^{\hat{t}+1}} \in A_2^{\hat{t}+1}$, $h^{\hat{t}+2} = \overline{h^{\hat{t}+1}}^{m_2^{\hat{t}+1}(m_1^{\hat{t}+1})}$, $\hat{s}_2(h_2^{\hat{t}+2}) = a_2^{\hat{t}+2,\ m_2^{\hat{t}+2}} \in A_2^{\hat{t}+2}$..., $h^{t+1} = \overline{h^t}^{m_2^t(m_1^t)}$.

Continue until $S_1^\Gamma(\hat{h}) = \bigcup_{m^{\hat{t}} \in \{1...\ n^{\hat{t}}\}} \cdots \bigcup_{m^T \in \{1...\ n^T\}} \{s_1 \in S_1^\Gamma(\hat{h}) |\ s_1(\hat{h}_1) = a_1^{\hat{t},\ m_1^{\hat{t}}} ...\ s_1(h_1^T) = a_1^{T,\ m_1^T}\}$ with $\hat{s}_2(\hat{h}_2) = a_2^{\hat{t},\ m_2^{\hat{t}}} \in A_2^{\hat{t}}$, $h^{\hat{t}+1} = \overline{\hat{h}}^{\hat{m}^{\hat{t}}(m^{\hat{t}})}$,..., $\hat{s}_2(h_2^{T-1}) = a_2^{T-1,\ m_2^{T-1}} \in A_2^{T-1}$, $h^T = \overline{h^{T-1}}^{m_2^{T-1}(m_2^{T-1})}$.

Now consider player 2's strategies other than the extreme ones.

Lemma 14. *For every* $s_2 \in S_2^\Gamma \backslash \{\underline{s}_2,\ \overline{s}_2\}$, $s_2(h_2^0) = a_2^{1,\ m_2^1}$ *and* $\hat{s}_1 \in BR^1(s_2)$, $\hat{s}_1(h^0) = a_1^{1,\ m_1^1}$, *if* $\overline{H}^{m_2^1(m_1^1)} \cap \{\overline{H}^{1(i)},\ \overline{H}^{n^1(j)}\} = \emptyset$, *then* $\{s_1 \in S_1^\Gamma |\ s_1(h^0) = \hat{s}_1(h^0)\} \subseteq BR^1(s_2)$.

Proof. Pick arbitrary $\tilde{s}_1 \in BR^1(\underline{s}_2)$ such that $\tilde{s}_1(h^0) = a_1^{1,\ i}$. As $\overline{H}^{m_2^1(m_1^1)} \cap \{\overline{H}^{1(i)},\ \overline{H}^{n^1(j)}\} = \emptyset$, $\overline{H}^{m_2^1(m_1^1)}$ are off the path of profile $(\tilde{s}_1,\ \underline{s}_2)$. Player 1 is indifferent among the choices on $\overline{H}^{m_2^1(m_1^1)} \cap H_1$. Let $\tilde{s}_1(\overline{H}^{m_2^1(m_1^1)} \cap H_1) = a_1^{2,\ n^2}$, then strategic complementarities imply that $\hat{s}_1 \vee \tilde{s}_1 \in BR^1(s_2)$. As $a_1^{1,\ i} \preceq \hat{s}_1(h^0)$, $(\hat{s}_1 \vee \tilde{s}_1)(h^0) = \hat{s}_1(h^0) \vee \tilde{s}_1(h^0) = a_1^{1,\ m_1^1}$ and $(\hat{s}_1 \vee \tilde{s}_1)(\overline{H}^{m_2^1(m_1^1)} \cap H_1) = a_1^{2,\ n^2}$.

Pick arbitrary $\tilde{s}_1' \in BR^1(\overline{s}_2)$ such that $\tilde{s}_1'(h^0) = a_1^{1,\ j}$. As $\overline{H}^{m_2^1(m_1^1)}$ is off the path of profile $(\tilde{s}_1',\ \overline{s}_2)$, player 1 must be indifferent among the choices on $\overline{H}^{m_2^1(m_1^1)} \cap H_1$. Let $\tilde{s}_1'(\overline{H}^{m_2^1(m_1^1)} \cap H_1) = a_1^{2,\ m_1^2}$, $m_1^2 \in \{1,...,\ n^2\}$, strategic complementarities imply that $(\hat{s}_1 \vee \tilde{s}_1) \wedge \tilde{s}_1' \in BR^1(s_2)$. As $\hat{s}_1(h^0) \preceq a_1^{1,\ j}$, $(\hat{s}_1 \vee \tilde{s}_1) \wedge \tilde{s}_1'(h^0) = (\hat{s}_1 \vee \tilde{s}_1)(h^0) \wedge \tilde{s}_1'(h^0) = a_1^{1,\ m_1^1}$ and $((\hat{s}_1 \vee \tilde{s}_1) \wedge \tilde{s}_1')(\overline{H}^{m_2^1(m_1^1)} \cap H_1) = a_1^{2,\ m_1^2}$.

Thus for arbitrary $m_1^2 \in \{1, ..., n^2\}$, there exists $s_1 \in BR^1(s_2)$ such that $s_1 \in S_1^\Gamma(\overline{H}^{m_2^1(m_1^1)})$ and $s_1(\overline{H}^{m_2^1(m_1^1)} \cap H_1) = a_1^{2,\ m_1^2}$.

For arbitrary $m_1^2 \in \{1,...,\ n^2\}$,..., $m_1^{t-1} \in \{1,...,\ n^{t-1}\}$, let $s_2(h_2^0) = a_1^{1,\ m_2^1}$, $s_2(\overline{H}^{m_2^1(m_1^1)} \cap H_2) = a_2^{2,\ m_2^2}$... $s_2(\overline{H}^{m_2^1(m_1^1),\ m_2^2(m_1^2),...,\ m_2^{t-3}(m_1^{t-3})} \cap H_2) = a_2^{t-2,\ m_2^{t-2}}$. Suppose there exists

$\hat{s}_1 \in BR^1(s_2)$ such that $\hat{s}_1 \in S_1^\Gamma(\overline{H}^{m_2^1(m_1^1),\ m_2^2(m_1^2),\dots,\ m_2^{t-2}(m_1^{t-2})})$ and $\hat{s}_1(\overline{H}^{m_2^1(m_1^1),\ m_2^2(m_1^2),\dots,\ m_2^{t-2}(m_1^{t-2})} \cap H_1) = a_1^{t-1,\ m_1^{t-1}}$. Suppose $s_2(\overline{H}^{m_2^1(m_1^1),\dots,\ m_2^{t-2}(m_1^{t-2})} \cap H_2) = a_2^{t-1,\ m_2^{t-1}}$.

Let $\tilde{s}_1 \in BR^1(\underline{s}_2)$ such that $\tilde{s}_1(h^0) = a_1^{1,\ i}$. As information sets $\overline{H}^{m_2^1(m_1^1)},\dots, \overline{H}^{m_2^1(m_1^1)\dots\ m_2^{t-1}(m_1^{t-1})}$ are off the path of profile $(\tilde{s}_1,\ \underline{s}_2)$, player 1 is indifferent among the choices on those information sets. Let $\tilde{s}_1(\overline{H}^{m_2^1(m_1^1)} \cap H_1) = \hat{s}_1(\overline{H}^{m_2^1(m_1^1)} \cap H_1),\dots,\ \tilde{s}_1(\overline{H}^{m_2^1(m_1^1),\dots,m_2^{t-2}(m_1^{t-2})} \cap H_1) = \hat{s}_1(\overline{H}^{m_2^1(m_1^1),\dots,m_2^{t-2}(m_1^{t-2})} \cap H_1)$ and $\tilde{s}_1(\overline{H}^{m_2^1(m_1^1),\dots,\ m_2^{t-1}(m_1^{t-1})} \cap H_1) = a_1^{t,\ n^t}$. $\underline{s}_2 \prec\ s_2$ and strategic complementarities implies that $\tilde{s}_1 \vee \hat{s}_1 \in\ BR^1(s_2)$. Thus $(\tilde{s}_1 \vee \hat{s}_1) \in S_1^\Gamma(\overline{H}^{m_2^1(m_1^1),\dots,\ m_2^{t-1}(m_1^{t-1})})$ and $(\tilde{s}_1 \vee \hat{s}_1)(\overline{H}^{m_2^1(m_1^1)\dots\ m_2^{t-1}(m_1^{t-1})} \cap H_1) = a_1^{t,\ n^t}$.

Let $\tilde{s}'_1 \in BR^1(\overline{s}_2)$ such that $\tilde{s}'(h^0) = a_1^{1,\ j}$. As information sets $\overline{H}^{m_2^1(m_1^1)},\dots, \overline{H}^{m_2^1(m_1^1)\dots\ m_2^{t-1}(m_1^{t-1})}$ are off the path of profile $(\tilde{s}'_1,\ \overline{s}_2)$, player 1 is indifferent among the choices on those information sets. Let $\tilde{s}'_1(\overline{H}^{m_2^1(m_1^1)} \cap H_1) = \hat{s}_1(\overline{H}^{m_2^1(m_1^1)} \cap H_1),\dots,\ \tilde{s}'_1(\overline{H}^{m_2^1(m_1^1),\dots,m_2^{t-2}(m_1^{t-2})} \cap H_1) = \hat{s}_1(\overline{H}^{m_2^1(m_1^1),\dots,m_2^{t-2}(m_1^{t-2})} \cap H_1)$ and $\tilde{s}'_1(\overline{H}^{m_2^1(m_1^1),\dots,\ m_2^{t-1}(m_1^{t-1})} \cap H_1) = a_1^{t,\ m_1^t}$, $m_1^t \in \{1,\dots,\ n^t - 1\}$. $\overline{s}_2 \succ\ s_2$ and strategic complementarities implies that $(\tilde{s}_1 \vee \hat{s}_1) \wedge \tilde{s}'_1 \in\ BR^1(s_2)$. Thus $(\tilde{s}_1 \vee \hat{s}_1) \wedge \tilde{s}'_1 \in S_1^\Gamma(\overline{H}^{m_2^1(m_1^1),\dots,\ m_2^{t-1}(m_1^{t-1})})$ and $(\tilde{s}_1 \vee \hat{s}_1) \wedge \tilde{s}'_1(\overline{H}^{m_2^1(m_1^1),\dots,\ m_2^{t-1}(m_1^{t-1})} \cap H_1) = a_1^{t,\ m_1^t}$ for arbitrary $m_1^t \in \{1,\dots,\ n^t - 1\}$.

Thus for arbitrary $m_1^2 \in \{1,\dots,\ n^2\},\dots,\ m_1^t \in \{1,\dots,\ n^t\}$, let $s_2(h_2^0) = a_1^{1,\ m_2^1}$, $s_2(\overline{H}^{m_2^1(m_1^1)} \cap H_2) = a_2^{2,\ m_2^2}\dots\ s_2(\overline{H}^{m_2^1(m_1^1)\dots\ m_2^{t-2}(m_1^{t-2})} \cap H_2) = a_2^{t-1,\ m_2^{t-1}}$, there exists $\hat{s}_1 \in BR^1(s_2)$ such that $\hat{s}_1(h^0) = a_1^{1,\ m_1^1}$, $\hat{s}_1(\overline{H}^{m_2^1(m_1^1)} \cap H_1) = a_1^{2,\ m_1^2},\dots,\ \hat{s}_1(\overline{H}^{m_2^1(m_1^1)\dots\ m_2^{t-1}(m_1^{t-1})} \cap H_1) = a_1^{t,\ m_1^t}$.

At $t = T$, for arbitrary $m_1^2 \in \{1,\dots,\ n^2\},\dots,\ m_1^T \in \{1,\dots,\ n^T\}$, let $s_2(h_2^0) = a_1^{1,\ m_2^1}$, $s_2(\overline{H}^{m_2^1(m_1^1)} \cap H_2) = a_2^{2,\ m_2^2}\dots,\ s_2(\overline{H}^{m_2^1(m_1^1),\dots,\ m_2^{T-2}(m_1^{T-2})} \cap H_2) = a_2^{T-1,\ m_2^{T-1}}$, there exists $\hat{s}_1 \in BR^1(s_2)$ such that $\hat{s}_1(h^0) = a_1^{1,\ m_1^1}$, $\hat{s}_1(\overline{H}^{m_2^1(m_1^1)} \cap H_1) = a_1^{2,\ m_1^2},\dots,\ \hat{s}_1(\overline{H}^{m_2^1(m_1^1),\dots,\ m_2^{T-1}(m_1^{T-1})} \cap H_1) = a_1^{T,\ m_1^T}$. As this denote one complete path of profile of $(\hat{s}_1,\ s_2)$, it is easy to see that $\{s_1 \in S_1^\Gamma |\ s_1(h^0) = a_1^{1,\ m_1^1},\ s_1(\overline{H}^{m_2^1(m_1^1)} \cap H_1) = a_1^{2,\ m_1^2},\dots,\ s_1(\overline{H}^{m_2^1(m_1^1),\dots,\ m_2^{T-1}(m_1^{T-1})} \cap H_1) =$

$a_1^{T,\ m_1^T}\} \subseteq BR^1(s_2)$.

Thus $S_1^\Gamma(\overline{H}^{m_2^1(m_1^1)}) = \bigcup_{m_1^2\in\{1,...,\ n^2\}} \cdots \bigcup_{m_1^T\in\{1,...,\ n^T\}}\{s_1 \in S_1^\Gamma|\ s_1(h^0) = a_1^{1,\ m_1^1},\ s_1(\overline{H}^{m_2^1(m_1^1)} \cap H_1) = a_1^{2,\ m_1^2}, ...,\ s_1(\overline{H}^{m_2^1(m_1^1),...,\ m_2^{T-1}(m_1^{T-1})} \cap H_1) = a_1^{T,\ m_1^T}\} \subseteq BR^1(s_2)$. ∎

It is easy to see that Lemma 5 applies to three general cases:

1. s_2 such that $s_2(h_2^0) \in A_2^1\backslash\{a_2^{1,\ 1},\ a_2^{1,\ n^1}\}$.

2. $s_2 \neq \underline{s}_2$ such that $s_2(h_2^0) = a_2^{1,\ 1}$ and there exists $s_1 \in BR^1(s_2)$, $s_1(h^0) \neq a_1^{1,\ i}$.

3. $s_2 \neq \overline{s}_2$ such that $s_2(h_2^0) = a_2^{1,\ n^1}$ and there exists $s_1 \in BR^1(s_2)$, $s_1(h^0) \neq a_1^{1,\ j}$.

Lemma 5 implies if Subgame $\overline{H}^{m_2^1(m_1^1)}$, different from Subgame $\overline{H}^{1(i)}$ and Subgame $\overline{H}^{n^1(j)}$, is reached on some best response path of s_2, then all subgames of Subgame $\overline{H}^{m_2^1(m_1^1)}$ that are consistent with s_2 can be reached on some best response path of s_2.

For player 2's strategy that start with actions within $A_2^1\backslash\{a_2^{1,\ 1},\ a_2^{1,\ n^1}\}$, the following structures on player 1's corresponding best response sets are satisfied under strategic complementarities assumption.

Theorem 5. *Consider arbitrary* $s_2 \in S_2^\Gamma$, $s_2(h_2^0) \in A_2^1\backslash\{a_2^{1,\ 1},\ a_2^{1,\ n^1}\}$,
a. $BR^1(s_2) = B$ *with* $B \subseteq \{S_1^\Gamma(\overline{H}^{m_1^1})|\ a_1^{1,\ i} \preceq a_1^{1,\ m_1^1} \preceq a_1^{1,\ j}\}$,
b. For arbitrary $\hat{s}_2 \in S_2^\Gamma$ *such that* $\hat{s}_2(h_2^0) \prec s_2(h_2^0)$ *and let* $s_2(h_2^0) = a_2^{1,\ m_2^1}$, $B(\hat{s}_2) \sqsubseteq B(inf S_2^\Gamma(\overline{H}^{m_2^1})) \sqsubseteq B(s_2) \sqsubseteq B(sup S_2^\Gamma(\overline{H}^{m_2^1}))$.

Proof. a. For arbitrary $\hat{s}_1 \in BR^1(s_2)$, $\hat{s}_1 \in \{s_1 \in S_1^\Gamma|\ s_1(h^0) = \hat{s}_1(h^0)\}$. Let $A(s_2) = \{s_1(h^0)|\ s_1 \in BR^1(s_2)\}$. Then $BR^1(s_2) \subseteq \bigcup_{a\in A(s_2)}\{s_1 \in S_1^\Gamma|\ s_1(h^0) = a\}$. Lemma 3 implies that $\bigcup_{a\in A(s_2)}\{s_1 \in S_1^\Gamma|\ s_1(h^0) = a\} \subseteq BR^1(s_2)$. Thus $BR^1(s_2) = \bigcup_{a\in A(s_2)}\{s_1 \in S_1^\Gamma|\ s_1(h^0) = a\}$.

b. Let $s_2(h_2^0) = a_2^{1,\ m_2^1}$ and $\hat{s}_2(h_2^0) = a_2^{1,\ \hat{m}_2^1}$. Form $\hat{s}_2'$ such that for all $h_2 \in \bigcup_{m_1^1 \in \{1,\ldots,\ n^1\}} \{h_2 \in H_2|\ h_2 \succeq \overline{H}^{\hat{m}_2^1(m_1^1)}\}$, $\hat{s}_2'(h_2) = \hat{s}_2(h_2)$ and for all $h_2 \in H_2 \backslash \bigcup_{m_1^1 \in \{1,\ldots,\ n^1\}} \{h_2 \in H_2|\ h_2 \succeq \overline{H}^{\hat{m}_2^1(m_1^1)}\}$, $\hat{s}_2'(h_2) = infA(h_2)$. Since information sets in $H_2 \backslash \bigcup_{m_1^1 \in \{1,\ldots,\ n^1\}} \{h_2 \in H_2|\ h_2 \succeq \overline{H}^{\hat{m}_2^1(m_1^1)}\}$ are off the paths of $(s_1,\ \hat{s}_2)$ for arbitrary $s_1 \in S_1^\Gamma$, $BR^1(\hat{s}_2) = BR^1(\hat{s}_2')$. Similarly, form s_2' such that for all $h_2 \in \bigcup_{m_1^1 \in \{1,\ldots,\ n^1\}} \{h_2 \in H_2|\ h_2 \succeq \overline{H}^{m_2^1(m_1^1)}\}$, $s_2'(h_2) = s_2(h_2)$ and for all $h_2 \in H_2 \backslash \bigcup_{m_1^1 \in \{1,\ldots,\ n^1\}} \{h_2 \in H_2|\ h_2 \succeq \overline{H}^{m_2^1(m_1^1)}\}$, $s_2'(h_2) = supA(h_2)$. $BR^1(s_2) = BR^1(s_2')$. Since $\hat{s}_2' \preceq s_2'$, $BR^1(\hat{s}_2') \sqsubseteq BR^1(s_2')$ and $BR^1(\hat{s}_2) \sqsubseteq BR^1(s_2)$. (a) implies $A(\hat{s}_2)$ and $A(s_2)$ exist. Pick arbitrary $\hat{s}_1 \in BR^1(\hat{s}_2)$ with $\hat{s}_1(h^0) = \hat{a} \in A(\hat{s}_2)$ and $s_1 \in BR^1(s_2)$ with $s_1(h^0) = a \in A(s_2)$, $\hat{s}_1 \vee s_1 \in BR^1(s_2)$ and $\hat{s}_1 \wedge s_1 \in BR^1(\hat{s}_2)$, in particular, $\hat{a} \vee a \in A(s_2)$ and $\hat{a} \wedge a \in A(\hat{s}_2)$. Thus $A(\hat{s}_2) \sqsubseteq A(s_2)$.

As $infS_2^\Gamma(\overline{H}^{m_2^1}) \prec s_2 \prec supS_2^\Gamma(\overline{H}^{m_2^1})$, strategic complementarities imply that $A(infS_2^\Gamma(\overline{H}^{m_2^1})) \sqsubseteq A(s_2) \sqsubseteq A(supS_2^\Gamma(\overline{H}^{m_2^1}))$. ∎

Now, consider player 2's strategy that assigns $a_2^{1,\ 1}$ at h_2^0. Some interesting common structure on the corresponding best response sets arise. They are discussed in Lemma 7 and 8 in which Lemma 7 can be considered as a special case covered by Lemma 8.

Lemma 15. *For arbitrary* $s_2 \in S_2^\Gamma(\overline{H}^{1(i)})$, $s_2(\overline{H}^{1(i)} \cap H_2) = a_2^{2,\ m_2^2} \in A_2^2 \backslash a_2^{2,\ 1}$,
a. If $BR^1(s_2) \cap S_1^\Gamma(\overline{H}^{1(i)}) \neq \emptyset$, *then* $BR^1(s_2) = S_1^\Gamma(\overline{H}^{1(i),\ m_2^2}) \cup A_1 \cup B$
b. If $BR^1(s_2) \cap S_1^\Gamma(\overline{H}^{1(i)}) = \emptyset$, *then* $BR^1(s_2) = B$
with $A_1 \subseteq \{S_1^\Gamma(\overline{H}^{1(i),\ m_2^2(m_1^2)}) | a^{2,\ m_1^2} \in A_1^2 \backslash a_1^{2,\ 1}\}$ *and* $B \subseteq \{S_1^\Gamma(\overline{H}^{1(m_1^1)})|\ a_1^{1,\ i} \prec a_1^{1,\ m_1^1} \preceq a_1^{1,\ j}\}$.

Proof. a. Pick arbitrary $\tilde{s}_1 \in BR^1(\overline{s}_2)$, as $\overline{H}^{1(i)}$ is off the path of $(\tilde{s}_1,\ \overline{s}_2)$, let $\tilde{s}_1(\overline{H}^{1(i)} \cap H_1) = a_1^{2,\ 1}$. There exists $s_1 \in BR^1(s_2)$, $s_1(h^0) = a_1^{1,\ i}$ as implied by the assumption. Strategic complementarities and $s_2 \prec \overline{s}_2$ implies that $s_1 \wedge \tilde{s}_1 \in BR^1(s_2)$, in particular,

$(s_1 \wedge \tilde{s}_1)(\overline{H}^{1(i)} \cap H_1) = a_1^{2,\ 1}$.

Thus there exists $s_1 \in BR^1(s_2)$ such that $s_1 \in S_1^\Gamma(\overline{H}^{1(i)})$ and $s_1(\overline{H}^{1(i)} \cap H_1) = a_1^{2,\ 1}$. And $\overline{H}^{1(i),\ \underline{m}_2^2}$ is reached on the path of $(s_1,\ s_2)$. Let $s_2(\overline{H}^{1(i),\ \underline{m}_2^2} \cap H_2) = a_2^{3,\ m_2^3}$.

Pick arbitrary $\tilde{s}_1 \in BR^1(\overline{s}_2)$, as $\overline{H}^{1(i),\ \underline{m}_2^2}$ is off the path of $(\tilde{s}_1,\ \overline{s}_2)$, let $\tilde{s}_1(\overline{H}^{1(i),\ \underline{m}_2^2} \cap H_1) = a_1^{3,\ 1}$. Strategic complementarities and $s_2 \prec \overline{s}_2$ implies that $s_1 \wedge \tilde{s}_1 \in BR^1(s_2)$, in particular, $(s_1 \wedge \tilde{s}_1)(\overline{H}^{1(i),\ \underline{m}_2^2} \cap H_1) = a_1^{3,\ 1}$. Pick arbitrary $\hat{s}_1 \in S_1^\Gamma(\overline{H}^{1(i),\ \underline{1}^2,\dots,\ \underline{1}^T}) \subseteq BR^1(\underline{s}_2)$. As $\overline{H}^{1(i),\ \underline{m}_2^2}$ is off the path of $(\hat{s}_1,\ \underline{s}_2)$, let $\hat{s}_1(\overline{H}^{1(i),\ \underline{m}_2^2} \cap H_1) = a_1^{3,\ m_1^3}$ with $m_1^3 \in \{1, \dots,\ n^3\}$.

Thus for arbitrary $a_1^{3,\ m_1^3} \in A_1^3$, there exists $s_1 \in BR^1(s_2)$ such that $s_1 \in S_1^\Gamma(\overline{H}^{1(i),\ \underline{m}_2^2})$ and $s_1(\overline{H}^{1(i),\ \underline{m}_2^2} \cap H_1) = a_1^{3,\ m_1^3}$. And $\overline{H}^{1(i),\ \underline{m}_2^2}$ is reached on the path of $(s_1,\ s_2)$. Let $s_2(\overline{H}^{1(i),\ \underline{m}_2^2, m_2^3(m_1^3)} \cap H_2) = a_2^{4,\ m_2^4}$.

For arbitrary $a_1^{3,\ m_1^3} \in A_1^3, \dots,\ a_1^{t-1,\ m_1^{t-1}} \in A_1^{t-1}$, suppose there exists $s_1 \in BR^1(s_2)$ such that $s_1 \in S_1^\Gamma(\overline{H}^{1(i),\ \underline{m}_2^2,\dots,\ m_2^{t-2}(m_1^{t-2})})$ and $s_1(\overline{H}^{1(i),\ \underline{m}_2^2,\dots,\ m_2^{t-2}(m_1^{t-2})} \cap H_1) = a_1^{t-1,\ m_1^{t-1}}$. Let $s_2(\overline{H}^{1(i),\ \underline{m}_2^2} \cap H_2) = a_2^{3,\ m_2^3}$, $\dots$, $s_2(\overline{H}^{1(i),\ \underline{m}_2^2,\dots,\ m_2^{t-3}(m_1^{t-3})} \cap H_2) = a_2^{t-2,\ m_2^{t-2}}$. Suppose $s_2(\overline{H}^{1(i),\ \underline{m}_2^2,\dots,\ m_2^{t-2}(m_1^{t-2})} \cap H_2) = a_2^{t-1,\ m_2^{t-1}}$.

Pick arbitrary $\tilde{s}_1 \in BR^1(\overline{s}_2)$, as $\overline{H}^{1(i),\ \underline{m}_2^2}$ and all $h \succ \overline{H}^{1(i),\ \underline{m}_2^2}$ are off the path of $(\tilde{s}_1,\ \overline{s}_2)$, let $\tilde{s}_1(\overline{H}^{1(i),\ \underline{m}_2^2} \cap H_1) = s_1(\overline{H}^{1(i),\ \underline{m}_2^2} \cap H_1), \dots,\ \tilde{s}_1(\overline{H}^{1(i),\ \underline{m}_2^2,\dots,\ m_2^{t-2}(m_1^{t-2})} \cap H_1) = s_1(\overline{H}^{1(i),\ \underline{m}_2^2,\dots,\ m_2^{t-2}(m_1^{t-2})} \cap H_1)$ and $\tilde{s}_1(\overline{H}^{1(i),\ \underline{m}_2^2,\dots,\ m_2^{t-1}(m_1^{t-1})} \cap H_1) = a_1^{t,\ 1}$. Strategic complementarities and $s_2 \prec \overline{s}_2$ implies that $s_1 \wedge \tilde{s}_1 \in BR^1(s_2)$, in particular, $(s_1 \wedge \tilde{s}_1)(\overline{H}^{1(i),\ \underline{m}_2^2,\dots,\ m_2^{t-1}(m_1^{t-1})} \cap H_1) = a_1^{t,\ 1}$. Pick arbitrary $\hat{s}_1 \in S_1^\Gamma(\overline{H}^{1(i),\ \underline{1}^2,\dots,\ \underline{1}^T}) \subseteq BR^1(\underline{s}_2)$. As $\overline{H}^{1(i),\ \underline{m}_2^2,\dots,\ m_2^{t-1}(m_1^{t-1})}$ is off the path of $(\hat{s}_1,\ \underline{s}_2)$, let $\hat{s}_1(\overline{H}^{1(i),\ \underline{m}_2^2,\dots,\ m_2^{t-1}(m_1^{t-1})} \cap H_1) = a_1^{t,\ m_1^t}$ with $a_1^{1,\ m_1^t} \in A_1^t$. Strategic complementarities and $\underline{s}_2 \prec s_2$ implies that $(s_1 \wedge \tilde{s}_1) \vee \hat{s}_1 \in BR^1(s_2)$. In particular,

$((s_1 \wedge \tilde{s}_1) \vee \hat{s}_1)(\overline{H}^{1(i),\ \underline{m}_2^2,...,\ m_2^{t-1}(m_1^{t-1})} \cap H_1) = a_1^{t,\ m_1^t}$.

Thus for arbitrary $a_1^{3,\ m_1^3} \in A_1^3$,..., $a_1^{t,\ m_1^t} \in A_1^t$, there exists $s_1 \in BR^1(s_2)$ such that $s_1 \in S_1^{\Gamma}(\overline{H}^{1(i),\ \underline{m}_2^2,...,\ m_2^{t-1}(m_1^{t-1})})$ and $s_1(\overline{H}^{1(i),\ \underline{m}_2^2,...,\ m_2^{t-2}(m_1^{t-2})} \cap H_1) = a_1^{t,\ m_1^t}$. Let $s_2(\overline{H}^{1(i),\ \underline{m}_2^2} \cap H_2) = a_2^{3,\ m_2^3}$, ..., $s_2(\overline{H}^{1(i),\ \underline{m}_2^2,...,\ m_2^{t-2}(m_1^{t-2})} \cap H_2) = a_2^{t-1,\ m_2^{t-1}}$.

Continue until $t = T$, for arbitrary $a_1^{3,\ m_1^3} \in A_1^3$,..., $a_1^{T,\ m_1^T} \in A_1^T$, let $s_2(\overline{H}^{1(i),\ \underline{m}_2^2} \cap H_2) = a_2^{3,\ m_2^3}$,..., $s_2(\overline{H}^{1(i),\ \underline{m}_2^2,...,\ m_2^{T-2}(m_1^{T-2})} \cap H_2) = a_2^{T-1,\ m_2^{T-1}}$, there exists $s_1 \in BR^1(s_2)$ such that $s_1(h^0) = a_1^{1,\ i}$, $s_1(\overline{H}^{1(i)} \cap H_1) = a_1^{2,\ 1}$,..., $s_1(\overline{H}^{1(i),...,\ m_2^{T-1}(m_1^{T-1})} \cap H_1) = a_1^{T,\ m_1^T}$. As this denote one complete path of profile of $(s_1,\ s_2)$, it is easy to see that $\{s_1 \in S_1^{\Gamma} |\ s_1(h^0) = a_1^{1,\ i},\ s_1(\overline{H}^{1(i)} \cap H_1) = a_1^{2,\ 1}, ...,\ s_1(\overline{H}^{1(i),...,\ m_2^{T-1}(m_1^{T-1})} \cap H_1) = a_1^{T,\ m_1^T}\} \subseteq BR^1(s_2)$.

Thus $S_1^{\Gamma}(\overline{H}^{1(i),\ \underline{m}_2^2}) = \bigcup_{m_1^3 \in \{1,...,\ n^2\}} \cdots \bigcup_{m_1^T \in \{1,...,\ n^T\}} \{s_1 \in S_1^{\Gamma} |\ s_1(h^0) = a_1^{1,\ m_1^1},\ s_1(\overline{H}^{m_2^1(m_1^1)} \cap H_1) = a_1^{2,\ m_1^2}, ...,\ s_1(\overline{H}^{m_2^1(m_1^1),...,\ m_2^{T-1}(m_1^{T-1})} \cap H_1) = a_1^{T,\ m_1^T}\} \subseteq BR^1(s_2)$.

A_1 can be proved similarly and B is implied by Lemma 3. ∎

Lemma 7 implies for s_2 that allows Subgame $\overline{H}^{1(i),\ \underline{m}_2^2}$ to be reached on the path, if Subgame $\overline{H}^{1(i)}$ is reached on some best response path of s_2, then every subgames of Subgame $\overline{H}^{1(i),\ \underline{m}_2^2}$ that are consistent with s_2 can be reached on some best response path of s_2. And if Subgame $\overline{H}^{1(i),\ m_2^2(m_1^2)}$ can be reached on some best response path of s_2, then every subgames of Subgame $\overline{H}^{1(i),\ m_2^2(m_1^2)}$ that are consistent with s_2 can be reached on some best response path of s_2. If subgame $\overline{H}^{1(m)}$ can be reached on some best response path of s_2, then every subgame of Subgame $\overline{H}^{1(m)}$ that are consistent with s_2 can be reached on some best response path of s_2.

Theorem 2 implies that for s_2 that allows Subgame $\overline{H}^{1(i),\underline{1}^2...\underline{1}^{\hat{t}-1},\ \underline{m}_2^{\hat{t}}}$ to be reached on the path, if Subgame $\overline{H}^{1(i)}$ is reached on some best response path of s_2, then every subgames of Subgame $\overline{H}^{1(i),\underline{1}^2...\underline{1}^{\hat{t}-1},\ \underline{m}_2^{\hat{t}}}$ that are consistent with s_2 can be reached on some best response path of s_2. And if Subgame $\overline{H}^{1(i),\underline{1}^2...\underline{1}^{\hat{t}-1},\ m_2^{\hat{t}}(m_1^{\hat{t}})}$ can be reached on some best response path of s_2, then every subgames of Subgame $\overline{H}^{1(i),\underline{1}^2...\underline{1}^{\hat{t}-1},\ m_2^{\hat{t}}(m_1^{\hat{t}})}$ that are consistent with s_2 can be reached on some best response path of s_2. If subgame $\overline{H}^{1(m)}$ can be reached on some best response path of s_2, then every subgame of Subgame $\overline{H}^{1(m)}$ that are consistent with s_2 can be reached on some best response path of s_2.

Once opponents' strategy reaches critical decision nodes, player i's best response must have certain structure in order for the overall game to exhibit strategic complementarities in player i.

Theorem 6. *For* $\hat{t} \in \{3, ...,\ T\}$ *and* $s_2 \in S_2^{\Gamma}(\overline{H}^{1(i),\underline{1}^2...\underline{1}^{\hat{t}-1}})$ *with* $s_2(\overline{H}^{1(i),\underline{1}^2...\underline{1}^{\hat{t}-1}} \cap H_2) = a_2^{\hat{t},\ m_2^{\hat{t}}} \in A_2^{\hat{t}} \backslash a_2^{\hat{t},\ 1}$,

a. If $BR^1(s_2) \cap \{s_1 \in S_1^{\Gamma} |\ s_1(h^0) = a_1^{1,\ i}\} \neq \emptyset$, *then* $BR^1(s_2) = S_1^{\Gamma}(\overline{H}^{1(i),\underline{1}^2\ ...\underline{1}^{\hat{t}-1},\ \underline{m}_2^{\hat{t}}}) \cup A_1 \cup A_2 \cup B$,

b. If $BR^1(s_2) \cap \{s_1 \in S_1^{\Gamma} |\ s_1(h^0) = a_1^{1,\ i}\} = \emptyset$, *then* $BR^1(s_2) = B$,

with $A_1 \subseteq \{S_1^{\Gamma}(\overline{H}^{1(i),\underline{1}^2\ ...\underline{1}^{\hat{t}-1},\ m_2^{\hat{t}}(m_1^{\hat{t}})}) |\ a_1^{\hat{t},\ m_1^{\hat{t}}} \in A_1^{\hat{t}} \backslash a_1^{\hat{t},\ 1}\}$, $A_2 \subseteq \{S_1^{\Gamma}(\overline{H}^{1(i),...,\ \underline{1}^{\tilde{t}-1},\ 1(m_1^{\tilde{t}})}) |\ \tilde{t} < \hat{t},\ a_1^{\tilde{t},\ m_1^{\tilde{t}}} \in A_1^{\tilde{t}} \backslash a_1^{\tilde{t},\ 1}\}$ *and* $B \in \{S_1^{\Gamma}(\overline{H}^{1(m_1^1)}) |\ a_1^{1,\ i} \prec a_1^{1,\ m_1^1} \preceq a_1^{1,\ j}\}$.

Proof. To prove the theorem, we want to prove the following claims:

a. if $BR^1(s_2) \cap \{s_1 \in S_1^{\Gamma} |\ s_1(h^0) = a_1^{1,\ i}\} \neq \emptyset$, then

(i). $S_1^{\Gamma}(\overline{H}^{1(i),\underline{1}^2\ ...\underline{1}^{\hat{t}-1},\ \underline{m}_2^{\hat{t}}}) \subseteq BR^1(s_2)$

(ii). If there exists $s_1 \in BR^1(s_2)$, $s_1 \in S_1^{\Gamma}(\overline{H}^{1(i),\underline{1}^2...\underline{1}^{\hat{t}-1}} \cap H_1)$ and $s_1(\overline{H}^{1(i),\underline{1}^2...\underline{1}^{\hat{t}-1}} \cap H_1) = a_1^{\hat{t},\ m_1^{\hat{t}}} \in A_1^{\hat{t}}$, $S_1^{\Gamma}(\overline{H}^{1(i),\underline{1}^2\ ...\underline{1}^{\hat{t}-1},\ m_2^{\hat{t}}(m_1^{\hat{t}})}) \subset BR^1(s_2)$

(iii). If there exists $\tilde{t} < \hat{t}$ such that there exists $s_1 \in BR^1(s_2) \cup BR^1(\underline{s}_2)$, $s_1 \in S_1^{\Gamma}(\overline{H}^{1(i),\ \underline{1}^2,...,\ \underline{1}^{\tilde{t}-1}})$ and $s_1(\overline{H}^{1(i),...,\ \underline{1}^{\tilde{t}-1}} \cap H_1) = a_1^{\tilde{t},\ m_1^{\tilde{t}}} \in A_1^{\tilde{t}} \backslash a_1^{\tilde{t},\ 1}$, then $S_1^{\Gamma}(\overline{H}^{1(i),...,\ \underline{1}^{\tilde{t}-1},\ 1(m_1^{\tilde{t}})}) \subset BR^1(s_2)$.

b. If there exists $s_1 \in BR^1(s_2)$, $s_1(h^0) = a_1^{1,\ m} \succ a_1^{1,\ i}$, then $\{s_1 \in S_1^\Gamma |\ s_1(h^0) = a_1^{1,\ m}\} \subset BR^1(s_2)$

a (i). Pick arbitrary $\tilde{s}_1 \in BR^1(\overline{s}_2)$ such that $\tilde{s}_1 \in S_1^\Gamma(\overline{H}^{n^1(j),...,\ \overline{n}^T})$, Lemma 4 indicates such $\tilde{s}_1$ exists. As $\overline{H}^{1(i)}$ are off the path of profile $(\tilde{s}_1,\ \overline{s}_2)$, assign $a_1^{2,\ 1}$ to $\tilde{s}_1(\overline{H}^{1(i)} \cap H_1)$. Pick arbitrary $\hat{s}_1 \in BR^1(s_2)$ such that $\hat{s}_1(h^0) = a_1^{1,\ i}$. Strategic complementarities imply that $\tilde{s}_1 \wedge \hat{s}_1 \in BR^1(s_2)$, in particular, $(\tilde{s}_1 \wedge \hat{s}_1)(h^0) = \tilde{s}_1(h^0) \wedge \hat{s}_1(h^0) = a_1^{1,\ i}$, $(\tilde{s}_1 \wedge \hat{s}_1)(\overline{H}^{1(i)} \cap H_1) = a_1^{2,\ 1}$. Thus there exists $s_1 \in BR^1(s_2)$ such that $s_1(h^0) = a_1^{1,\ i}$ and $s_1(\overline{H}^{1(i)} \cap H_1) = a_1^{2,\ 1}$.

Suppose there exists $s_1 \in BR^1(s_2)$, $s_1(h^0) = a_1^{1,\ i}$, $s_1(\overline{H}^{1(i)} \cap H_1) = a_1^{2,\ 1}$... $s_1(\overline{H}^{1(i),...,\ \underline{1}^{\hat{t}-2}} \cap H_1) = a_1^{\hat{t}-1,\ 1}$, pick arbitrary $\tilde{s}_1 \in BR^1(\overline{s}_2)$ such that $\tilde{s}_1 \in S_1^\Gamma(\overline{H}^{n^1(j),...,\ \overline{n}^T})$ and $\tilde{s}_1(\overline{H}^{1(i),...,\ \underline{1}^{\hat{t}-1}} \cap H_1) = a_1^{\hat{t},\ 1}$. Strategic complementarities imply that $\tilde{s}_1 \wedge s_1 \in BR^1(s_2)$, in particular, $(\tilde{s}_1 \wedge s_1)(\overline{H}^{1(i),...,\ \underline{1}^{\hat{t}-1}} \cap H_1) = a_1^{\hat{t},\ 1}$.

Thus given $s_2(\overline{H}^{1(i),\underline{1}^2...\underline{1}^{\hat{t}-1}} \cap H_2) = a_2^{\hat{t},\ m_2^{\hat{t}}}$, there exists $s_1 \in BR^1(s_2)$, $s_1 \in S_1^\Gamma(\overline{H}^{1(i),...,\ \underline{1}^{\hat{t}-1},\ \underline{m}_2^{\hat{t}}})$.

Pick arbitrary $\tilde{s}_1 \in S_1^\Gamma(\overline{H}^{n^1(j),...,\ \overline{n}^T}) \subseteq BR^1(\overline{s}_2)$. As $\overline{H}^{1(i),...,\ \underline{1}^{\hat{t}-1},\ \underline{m}_2^{\hat{t}}}$ are off the path of profile $(\tilde{s}_1,\ \overline{s}_2)$, let $\tilde{s}_1(\overline{H}^{1(i),...,\ \underline{1}^{\hat{t}-1},\ \underline{m}_2^{\hat{t}}} \cap H_1) = a_1^{\hat{t}+1,\ 1}$. Strategic complementarities imply that $\tilde{s}_1 \wedge s_1 \in BR^1(s_2)$, in particular, $\tilde{s}_1 \wedge s_1(\overline{H}^{1(i),...,\ \underline{1}^{\hat{t}-1},\ \underline{m}_2^{\hat{t}}} \cap H_1) = a_1^{\hat{t}+1,\ 1}$.

Let $\hat{s}_1 \in BR^1(\underline{s}_1)$ such that $\hat{s}_1 \in S_1^\Gamma(\overline{H}^{1(i),...,\ \underline{1}^{T-1},\ \underline{1}^T})$, Lemma 4 indicates such $\hat{s}_1$ exists. $\overline{H}^{1(i),...,\underline{1}^{\hat{t}-1},\ \underline{m}_2^{\hat{t}}}$ are off the path of $(\hat{s}_1,\ \underline{s}_2)$, assign $\hat{s}_1(\overline{H}^{1(i),...,\ \underline{1}^{\hat{t}-1},\ \underline{m}_2^{\hat{t}}} \cap H_1)$ with $a_1^{\hat{t}+1,\ m_1^{\hat{t}+1}}$ for $m_1^{\hat{t}+1} \in \{1,...,\ n^{\hat{t}+1}\}$. Strategic complementarities imply that $(\tilde{s}_1 \wedge s_1) \vee \hat{s}_1 \in BR^1(s_2)$, $(\tilde{s}_1 \wedge s_1) \vee \hat{s}_1 \in S_1^\Gamma(\overline{H}^{1(i),...,\ \underline{1}^{\hat{t}-1},\ \underline{m}_2^{\hat{t}}})$ and $(\tilde{s}_1 \wedge s_1) \vee \hat{s}_1(\overline{H}^{1(i),...,\ \underline{1}^{\hat{t}-1},\ \underline{m}_2^{\hat{t}}} \cap H_1) = a_1^{\hat{t}+1,\ m_1^{\hat{t}+1}}$.

Thus for every $m_1^{\hat{t}+1} \in \{1,...,\ n^{\hat{t}+1}\}$, there exists $s_1 \in BR^1(s_2)$, $s_1 \in S_1^\Gamma(\overline{H}^{1(i),...,\ \underline{1}^{\hat{t}-1},\ \underline{m}_2^{\hat{t}}})$ and $s_1(\overline{H}^{1(i),...,\ \underline{1}^{\hat{t}-1},\ \underline{m}_2^{\hat{t}}} \cap H_1) = a_1^{\hat{t}+1,\ m_1^{\hat{t}+1}}$.

Suppose for arbitrary $m_1^{\hat{t}+1} \in \{1...\ n^{t+1}\}$... $m_1^{\hat{t}+s-1} \in \{1...\ n^{\hat{t}+s-1}\}$, let $s_2(\overline{H}^{1(i)...\ \underline{m}^{\hat{t}}} \cap$

$H_2) = a_2^{\hat{t}+1,\ m_2^{\hat{t}+1}} \ldots\ s_2(\overline{H}^{1(i)\ldots\ \underline{m}_2^{\hat{t}},\ m_2^{\hat{t}+1}(m_1^{\hat{t}+1})\ldots m_2^{\hat{t}+s-2}(m_1^{\hat{t}+s-2})} \cap H_2) = a_2^{\hat{t}+s-1,\ m_2^{\hat{t}+s-1}}$, there exists $s_1 \in BR^1(s_2)$ such that $s_1 \in S_1^{\Gamma}(\overline{H}^{1(i),\ldots,\ \underline{1}^{\hat{t}-1},\underline{m}_2^{\hat{t}},\ m_2^{\hat{t}+1}(m_1^{\hat{t}+1})\ldots\ m_2^{\hat{t}+s-1}(m_1^{\hat{t}+s-1})})$.

Pick arbitrary $\tilde{s}_1 \in S_1^{\Gamma}(\overline{H}^{n^1(j),\ldots,\ \overline{n}^T}) \subseteq BR^1(\overline{s}_2)$. As $\overline{H}^{1(i)\ldots\ \underline{1}^{\hat{t}-1},\underline{m}_2^{\hat{t}}}$, $\overline{H}^{1(i)\ldots\ \underline{m}_2^{\hat{t}},\ m_2^{\hat{t}+1}(m_1^{\hat{t}+1})}$,..., $\overline{H}^{1(i)\ldots\ \underline{m}_2^{\hat{t}},\ m_2^{\hat{t}+1}(m_1^{\hat{t}+1})\ldots\ m_2^{\hat{t}+s-1}(m_1^{\hat{t}+s-1})}$ are off the path of profile $(\tilde{s}_1, \overline{s}_2)$, assign $\tilde{s}_1$ with the following action on those information sets: $\tilde{s}_1(\overline{H}^{1(i)\ldots\ \underline{m}_2^{\hat{t}}} \cap H_1) = s_1(\overline{H}^{1(i)\ldots\underline{m}_2^{\hat{t}}} \cap H_1)$,..., $\tilde{s}_1(\overline{H}^{1(i)\ldots m_2^{\hat{t}+s-2}(m_1^{\hat{t}+s-2})} \cap H_1) = s_1(\overline{H}^{1(i)\ldots m_2^{\hat{t}+s-2}(m_1^{\hat{t}+l-2})} \cap H_1)$ and $\tilde{s}_1(\overline{H}^{1(i)\ldots m_2^{\hat{t}+s-1}(m_1^{\hat{t}+s-1})} \cap H_1) = a_1^{\hat{t}+s,\ 1}$. Strategic complementarities implies that $\tilde{s}_1 \wedge s_1 \in BR^1(s_2)$, in particular, $\tilde{s}_1 \wedge s_1(\overline{H}^{1(i)\ldots\ m_2^{\hat{t}+s-1}(m_1^{t+s-1})} \cap H_1) = a_1^{t+s,\ 1}$.

Pick arbitrary $\hat{s}_1 \in S_1^{\Gamma}(\overline{H}^{1(i),\ldots,\ \underline{1}^{T-1},\ \underline{1}^T}) \subseteq BR^1(\underline{s}_2)$. As $\overline{H}^{1(i)\ldots\ \underline{1}^{\hat{t}-1},\underline{m}_2^{\hat{t}}}$, $\overline{H}^{1(i),\ldots\ \underline{m}_2^{\hat{t}},\ m_2^{\hat{t}+1}(m_1^{\hat{t}+1})}$,..., $\overline{H}^{1(i)\ldots,\ m_2^{\hat{t}+1}(m_1^{\hat{t}+1}),\ldots,\ m_2^{\hat{t}+s-1}(m_1^{\hat{t}+s-1})}$ are off the path of profile $(\hat{s}_1, \underline{s}_2)$, player 1 is indifferent among the choices on those two information sets. Assign $\hat{s}_1$ with the following actions on those information sets, in particular, let $\hat{s}_1(\overline{H}^{1(i)\ldots\ \underline{m}_2^{\hat{t}}} \cap H_1) = s_1(\overline{H}^{1(i)\ldots\ \underline{m}_2^{\hat{t}}} \cap H_1)$,..., $\hat{s}_1(\overline{H}^{1(i)\ldots\ m_2^{\hat{t}+s-2}(m_1^{\hat{t}+s-2})} \cap H_1) = s_1(\overline{H}^{1(i)\ldots\ m_2^{\hat{t}+s-2}(m_1^{\hat{t}+s-2})} \cap H_1)$ and $\hat{s}_1(\overline{H}^{1(i)\ldots\ m_2^{\hat{t}+s-1}(m_1^{\hat{t}+s-1})} \cap H_1) = a_1^{\hat{t}+s,\ m_1^{\hat{t}+s}}$, $m_1^{\hat{t}+s} \in \{1, \ldots,\ n^{\hat{t}+s} - 1\}$. Strategic complementarities implies that $(\tilde{s}_1 \wedge s_1) \vee \hat{s}_1 \in BR^1(s_2)$, in particular, $((\tilde{s}_1 \wedge s_1) \vee \hat{s}_1)(\overline{H}^{1(i)\ldots\ m_2^{\hat{t}+s-1}(m_1^{\hat{t}+s-1})} \cap H_1) = a_1^{\hat{t}+s,\ m_1^{\hat{t}+s}}$.

Thus for arbitrary $m_1^{\hat{t}+1} \in \{1\ldots\ n^{\hat{t}+1}\}\ldots\ m_1^{\hat{t}+s} \in \{1\ldots\ n^{\hat{t}+s}\}$, there exists $s_1 \in BR^1(s_2)$ such that $s_1 \in S_1^{\Gamma}(\overline{H}^{1(i),\ldots,\ \underline{1}^{\hat{t}-1},\ \underline{m}_2^{\hat{t}}})$, $s_1(\overline{H}^{\underline{1}^1\ldots\ \underline{m}_2^{\hat{t}}} \cap H_1) = a_1^{\hat{t}+1,\ m_1^{\hat{t}+1}}\ \ldots\ s_1(\overline{H}^{\underline{1}^1,\ldots,m_2^{\hat{t}+s-1}(m_1^{\hat{t}+s-1})} \cap H_1) = a_1^{\hat{t}+s,\ m_1^{\hat{t}+s}}$.

Continue until $\hat{t} + s = T$, for arbitrary $m_1^{\hat{t}+1} \in \{1\ldots\ n^{\hat{t}+1}\}$,..., $m_1^T \in \{1\ldots\ n^T\}$, there exists $s_1 \in BR^1(s_2)$ such that $s_1 \in S_1^{\Gamma}(\overline{H}_2^{1(i),\ldots,\ \underline{1}^{\hat{t}-1},\ \underline{m}_2^{\hat{t}},\ m_2^{\hat{t}+1}(m_1^{\hat{t}+1}),\ldots,m_2^T(m_1^T)})$. Since each $\overline{H}^{1(i),\ldots,\ m_2^T(m_1^T)}$ is an ending node, $S_1^{\Gamma}(\overline{H}^{1(i),\ldots,\ \underline{1}^{\hat{t}-1},\ \underline{m}_2^{\hat{t}},\ m_2^{\hat{t}+1}(m_1^{\hat{t}+1}),\ldots,m_2^T(m_1^T)}) \subset BR^1(s_2)$.

Thus $\bigcup_{m_1^{\hat{t}+1} \in \{1\ldots\ n^{\hat{t}+1}\}} \cdots \bigcup_{m_1^T \in \{1\ldots\ n^T\}} \{s_1 \in S_1^{\Gamma}(\hat{h}) |\ s_1(\hat{h}_1) = a_1^{\hat{t}+1,\ m_1^{\hat{t}+1}} \ldots\ s_1(h_1^T) =$

$a_1^{T,\ m_1^T}\} \subset BR^1(s_2)$ with $\hat{h} = \overline{H}^{1(i),...,\ \underline{1}^{\hat{t}-1},\ \underline{m}_2^{\hat{t}}}$. $s_2(\hat{h}_2) = a_2^{\hat{t}+1,\ m_2^{\hat{t}+1}} \in A_2^{\hat{t}+1}$, $h^{\hat{t}+2} = \overline{\hat{h}}^{m_2^{\hat{t}+1}(m_1^{\hat{t}+1})}$,..., $s_2(h_2^{T-1}) = a_2^{T-1,\ m_2^{T-1}} \in A_2^{T-1}$, $h^T = \overline{h^{T-1}}^{m_2^{T-1}(m_2^{T-1})}$. Thus $S_1^\Gamma(\overline{H}^{1(i),...,\ \underline{1}^{\hat{t}-1},\ \underline{m}_2^{\hat{t}}}) \subset BR^1(s_2)$.

(ii) Proved similarly as Theorem 1(iii)

(iii). From Theorem 1(iii), $S_1^\Gamma(\overline{H}^{1(i),...,\ \underline{1}^{\tilde{t}-1},\ 1(m_1^{\tilde{t}})}) \subset BR^1(\underline{s}_2)$.

Pick arbitrary $\hat{s}_1 \in S_1^\Gamma(\overline{H}^{1(i),...,\ \underline{1}^{\tilde{t}-1},\ 1(m_1^{\tilde{t}})})$ and $\tilde{s}_1 = inf S_1^\Gamma(\overline{H}^{1(i),...,\ \underline{1}^{\hat{t}-1},\ \underline{m}_2^{\hat{t}}})$. $\underline{s}_2 \prec s_2$ and strategic complementarities imply that $\hat{s}_1 \vee \tilde{s}_1 \in BR^1(s_2)$, in particular, $\hat{s}_1 \vee \tilde{s}_1 = \hat{s}_1$. Thus $S_1^\Gamma(\overline{H}^{1(i),...,\ \underline{1}^{\tilde{t}-1},\ 1(m_1^{\tilde{t}})}) \subset BR^1(s_2)$.

b. It is implied by Lemma 3. ▮

Corollary 9. *Consider $s_2 \in S_2^\Gamma$ with $s_2(h^0) = a_2^{1,\ n^1}$,*
a. For $s_2 \in S_2^\Gamma(\overline{H}^{n^1(j)})$, $s_2(\overline{H}^{n^1(j)} \cap H_2) = a_2^{2,\ m_2^2} \in A_2^2 \backslash a_2^{2,\ n^2}$, if there exists $s_1 \in BR^1(s_2)$, $s_1(h^0) = a_1^{1,\ j}$, then $S_1^\Gamma(\overline{H}^{n^1(j),\ \overline{m}_2^2}) \subseteq BR^1(s_2)$ and if there exists $s_1 \in BR^1(s_2)$, $s_1(\overline{H}^{n^1(j)} \cap H_1) = a_1^{2,\ m_1^2} \in A_1^2$, then $S_1^\Gamma(\overline{H}^{n^1(j),\ m_2^2(m_1^2)}) \subset BR^1(s_2)$.
b. For $\hat{t} \in \{3, ..., T\}$ and $s_2 \in S_2^\Gamma(\overline{H}^{n^1(j),\overline{n}^2...\overline{n}^{\hat{t}-1}})$ with $s_2(\overline{H}^{n^1(j),\overline{n}^2...\overline{n}^{\hat{t}-1}} \cap H_2) = a_2^{\hat{t},\ m_2^{\hat{t}}} \in A_2^{\hat{t}} \backslash a_2^{\hat{t},\ n^{\hat{t}}}$, i. if $BR^1(s_2) \cap \{s_1 \in S_1^\Gamma | s_1(h^0) = a_1^{1,\ j}\} \neq \emptyset$, then $BR^1(s_2) = S_1^\Gamma(\overline{H}^{n^1(j),\overline{n}^2\ ...\overline{n}^{\hat{t}-1},\ \overline{m}_2^{\hat{t}}}) \cup A_1 \cup A_2 \cup B$ ii. If $BR^1(s_2) \cap \{s_1 \in S_1^\Gamma | s_1(h^0) = a_1^{1,\ j}\} = \emptyset$, then $BR^1(s_2) = B$,
with $A_1 \subseteq \{S_1^\Gamma(\overline{H}^{n^1(j),\overline{n}^2\ ...\overline{n}^{\hat{t}-1},\ m_2^{\hat{t}}(m_1^{\hat{t}})}) |\ a_1^{\hat{t},\ m_1^{\hat{t}}} \in A_1^{\hat{t}}\}$, $A_2 \subseteq \{S_1^\Gamma(\overline{H}^{n^1(j),...,\ \overline{n}^{\tilde{t}-1},\ n^{\tilde{t}}(m_1^{\tilde{t}})}) |\ \tilde{t} < \hat{t},\ a_1^{\tilde{t},\ m_1^{\tilde{t}}} \in A_1^{\tilde{t}} \backslash a_1^{\tilde{t},\ n^1}\}$ and $B \in \{S_1^\Gamma(\overline{H}^{n^1(m_1^1)}) |\ a_1^{1,\ i} \prec a_1^{1,\ m_1^1} \preceq a_1^{1,\ j}\}$

Proof. Proved similarly as Theorem 3. ▮

Theorem 7. *For all s_2 such that $s_2(h_2^0) = a_2^{1,\ 1}$,*
i. $B(\underline{s}_2) \sqsubseteq B(s_2) \sqsubseteq B(supS_2^\Gamma(\overline{H}^{\underline{1}}))$,
ii. if $BR^1(s_2) \cap \{s_1 \in S_1^\Gamma |\ s_1(h^0) = a_1^{1,\ i}\} \neq \emptyset$, $BR^1(\underline{s}_2) \subseteq BR^1(s_2)$ and moreover, if

$B(\underline{s}_2) \neq \emptyset$, $BR^1(\underline{s}_2) \backslash B(\underline{s}_2) = BR^1(s_2) \backslash B(s_2)$.

Proof. (i). Directly obtained from the definition of strategic complementarities.

(ii). Let $s_1 = inf S_1^{\Gamma}(\overline{H}^{1(i),\ \underline{1}^2,\dots,\ \underline{1}^T})$, Theorem 3 implies $s_1 \in BR^1(s_2)$. Pick arbitrary $s_1' \in BR^1(\underline{s}_2)$, strategic complementarities and $s_2 \prec s_2'$ implies $s_1 \vee s_1' \in BR^1(s_2)$. As $s_1 \vee s_1' = s_1'$, $BR^1(\underline{s}_2) \subseteq BR^1(s_2)$. Thus $BR^1(\underline{s}_2) \cap \{s_1 \in S_1^{\Gamma} |\ s_1(h^0) = a_1^{1,\ i}\} \subseteq BR^1(s_2) \cap \{s_1 \in S_1^{\Gamma} |\ s_1(h^0) = a_1^{1,\ i}\}$

Assuming $B(\underline{s}_2) \neq \emptyset$, pick arbitrary $\{s_1 \in S_1^{\Gamma} |\ s_1(h^0) = a_1^{1,\ m_1^1}\} \subset B(\underline{s}_2)$. Let $\hat{s}_1 = sup\{s_1 \in S_1^{\Gamma} |\ s_1(h^0) = a_1^{1,\ m_1^1}\}$, then $\hat{s}_1 \in BR^1(\underline{s}_2)$. Pick arbitrary $\tilde{s}_1 \in BR^1(s_2) \cap \{s_1 |\ s_1(h^0) = a_1^{1,\ i}\}$. $s_2 \succeq \underline{s}_2$ and strategic complementarity implies that $\hat{s}_1 \wedge \tilde{s}_1 \in BR^1(\underline{s}_2)$, in particular, $\tilde{s}_1 = \hat{s}_1 \wedge \tilde{s}_1$. Thus $BR^1(s_2) \cap \{s_1 \in S_1^{\Gamma} |\ s_1(h^0) = a_1^{1,\ i}\} \subseteq BR^1(\underline{s}_2)$ and $BR^1(s_2) \cap \{s_1 \in S_1^{\Gamma} |\ s_1(h^0) = a_1^{1,\ i}\} \subseteq BR^1(\underline{s}_2) \cap \{s_1 \in S_1^{\Gamma} |\ s_1(h^0) = a_1^{1,\ i}\}$. Thus $BR^1(\underline{s}_2) \backslash B(\underline{s}_2) = BR^1(s_2) \backslash B(s_2)$. ∎

Corollary 10. *For all* $s_2 \in \{s_2 |\ s_2(h_2^0) = a_2^{1,\ n^1}\}$, $B(inf S_2^{\Gamma}(\overline{H}^{\underline{n}^1})) \sqsubseteq B(s_2) \sqsubseteq B(\overline{s}_2)$ *and if* $BR^1(s_2) \cap \{s_1 \in S_1^{\Gamma} |\ s_1(h^0) = a_1^{1,\ j}\} \neq \emptyset$, $BR^1(\overline{s}_2) \subseteq BR^1(s_2)$ *and moreover, if* $B(\overline{s}_2) \neq \emptyset$, $BR^1(s_2) \backslash B(s_2) = BR^1(\overline{s}_2) \backslash B(\overline{s}_2)$.

Proof. Similarly as Theorem 4.∎

Theorem 8. *Under differential payoff to outcome assumption, the only two-player multi-stage* ($t > 1$) *game that can exhibit strategic complements is the class of two-stage games with first stage game being a* 2×2 *game.*

Proof. Consider player 1 only, the cases for player 2 is similar. The proof consists of two parts. First, we prove that the class of two-stage games with first stage game being a 2×2 game can exhibits strategic complements. Second, we prove that the rest of the classes cannot.

Suppose we are considering a two-stage game with the first stage game being a 2×2 game and second stage game being a $m \times n$ game, $m,\ n \geq 2$. The way to show it can exhibits strategic complements is similar to the first part in Chapter 2 when we characterize strategic complements for the two-stage 2×2 game. Under differential payoff to outcome assumption, it is easy to check that strategic complements can be generated in the following scenario.

- $a_1^{2,\ 1}$ in subgame 1 dominates $\{a_1^{2,\ 2},\ a_1^{2,\ 3}, ...,\ a_1^{2,\ m}\}$ in subgame 1 and $\{a_1^{2,\ 1},\ a_1^{2,\ 2}, ...,\ a_1^{2,\ m}\}$ in subgame 3, and $a_1^{2,\ m}$ in subgame 2 dominates $\{a_1^{2,\ 1},\ a_1^{2,\ 2}, ...,\ a_1^{2,\ m-1}\}$ in subgame 2 and $\{a_1^{2,\ 1},\ a_1^{2,\ 2}, ...,\ a_1^{2,\ m}\}$ in subgame 4.

- $a_1^{2,\ 1}$ in subgame 1 dominates $\{a_1^{2,\ 2},\ a_1^{2,\ 3}, ...,\ a_1^{2,\ m}\}$ in subgame 1 and $\{a_1^{2,\ 1},\ a_1^{2,\ 2}, ...,\ a_1^{2,\ m}\}$ in subgame 3, and $a_1^{2,\ m}$ in subgame 4 dominates $\{a_1^{2,\ 1},\ a_1^{2,\ 2}, ...,\ a_1^{2,\ m-1}\}$ in subgame 4 and $\{a_1^{2,\ 1},\ a_1^{2,\ 2}, ...,\ a_1^{2,\ m}\}$ in subgame 2.

- $a_1^{2,\ 1}$ in subgame 3 dominates $\{a_1^{2,\ 2},\ a_1^{2,\ 3}, ...,\ a_1^{2,\ m}\}$ in subgame 3 and $\{a_1^{2,\ 1},\ a_1^{2,\ 2}, ...,\ a_1^{2,\ m}\}$ in subgame 1, and $a_1^{2,\ m}$ in subgame 4 dominates $\{a_1^{2,\ 1},\ a_1^{2,\ 2}, ...,\ a_1^{2,\ m-1}\}$ in subgame 4 and $\{a_1^{2,\ 1},\ a_1^{2,\ 2}, ...,\ a_1^{2,\ m}\}$ in subgame 2.

Secondly, consider multi-stage games that does not belong to the classes discussed above.

First, consider a multi-stage game with the first stage being a $m \times n$ game, $m \geq 2$ and $n > 3$. Pick arbitrary $s_2 \in S_2^{\Gamma}$ such that $s_2(h_2^0) \in A_2^1 \backslash \{a_2^{1,\ 1},\ a_2^{1,\ n}\}$, then Theorem 5(1) indicates that more than one terminal nodes can be reached on the best response path. Thus differential payoffs to outcomes assumption will be violated.

Second, consider a multi-stage game with $t > 2$. Pick arbitrary $s_2 \in S_2^{\Gamma}$ such that $s_2(h_2^0) = a_2^{1,\,1}$ and $s_2(h_2^2) = A_2^2 \backslash a_2^{1,\,1}$. Lemma 15 indicates more than one terminal nodes can be reached on the best response path with respect to s_2, thus differential payoffs to outcomes assumption will be violated. ∎

4.3 Example

Now consider the general two-stage game below.

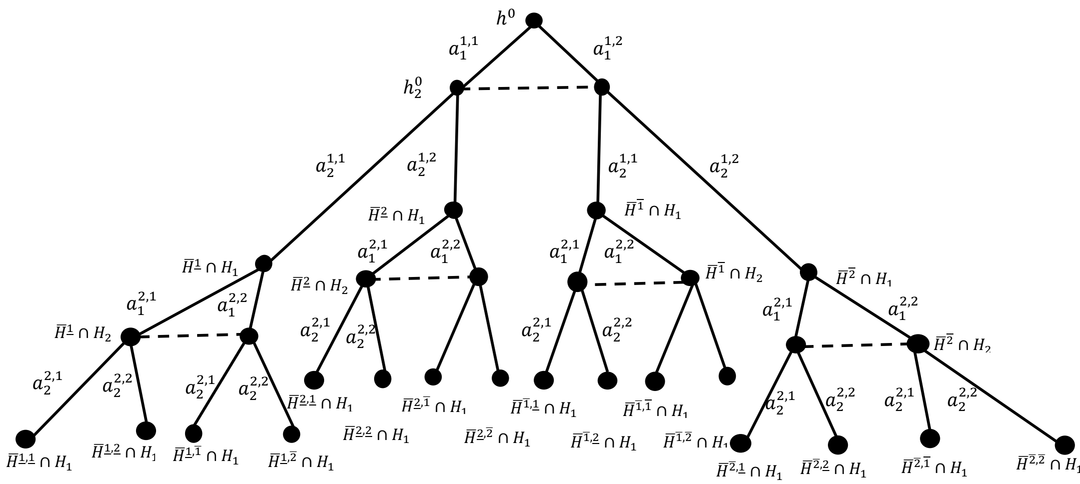

Figure 12: $2 - stage\ extensive\ form$

i. Let $inf_{s_1 \in BR^1(\underline{s}_2)} s_1(h^0) = a_1^{1,\,1}$ and $sup_{s_1 \in BR^1(\bar{s}_2)} s_1(h^0) = a_1^{1,\,1}$.

Thus for all $s_2 \in S_2^{\Gamma}$, $BR^1(s_2) \subseteq \{s_1 \in S_1^{\Gamma} |\ s_1(h_2^0) = a_1^{1,\,1}\}$.

Consider $s_2 \in S_2^{\Gamma}$ such that $s_2(h_2^0) = a_2^{1,\,1}$, For $\underline{s}_2 \in S_2^\Gamma$, Theorem 1 implies that $BR^1(\underline{s}_2) = S_1^\Gamma(\overline{H}^{\underline{1}^1,\ \underline{1}^2}) \cup A_1$ in which $A_1 \in \{\emptyset,\ S_1^\Gamma(\overline{H}^{\underline{1}^1,\ \overline{1}^2})\}$.

For $s_2 \in S_2^\Gamma(\overline{H}^{\underline{1},\ \underline{1}}) \setminus \underline{s}_2$, $BR^1(s_2) = BR^1(\underline{s}_2)$

For $s_2 \in S_2^\Gamma(\overline{H}^{\underline{1}})$ and $s_2(\overline{H}^{\underline{1}} \cap H_2) = a_2^{2,\ 2}$, Theorem 4 implies $BR^1(\underline{s}_2) \subseteq BR^1(s_2)$.

If $A_1 = \emptyset$, then either $BR^1(\underline{s}_2) = BR^1(s_2)$ or $BR^1(s_2) = S_1^\Gamma(\overline{H}^{\underline{1}})$. Then the corresponding payoffs are $r_1^{1,\ 1}(> r_1^{1,\ 3}), r_1^{1,\ 2}(\geq r_1^{1,\ 4}) > r_1^{3,\ 1},\ r_1^{3,\ 2},\ r_1^{3,\ 3},\ r_1^{3,\ 4}$.

If $A_1 = S_1^\Gamma(\overline{H}^{\underline{1}^1,\ \overline{1}^2})$, then $BR^1(\underline{s}_2) = BR^1(s_2)$. The corresponding payoffs are $r_1^{1,\ 1}(= r_1^{1,\ 3}), r_1^{1,\ 2}(= r_1^{1,\ 4}) > r_1^{3,\ 1},\ r_1^{3,\ 2},\ r_1^{3,\ 3},\ r_1^{3,\ 4}$

For $\overline{s}_2 \in S_2^\Gamma$, Corr 1 implies that $BR^1(\overline{s}_2) = S_1^\Gamma(\overline{H}^{\underline{2}^1,\ \overline{2}^2}) \cup A_1$ in which $A_1 \in \{\emptyset,\ S_1^\Gamma(\overline{H}^{\underline{2},\ \underline{2}})\}$.

For $s_2 \in S_2^\Gamma(\overline{H}^{\underline{2},\ \overline{2}}) \setminus \overline{s}_2$, $BR^1(s_2) = BR^1(\overline{s}_2)$

For $s_2 \in S_2^\Gamma(\overline{H}^{\underline{2}})$ and $s_2(\overline{H}^{\underline{2}} \cap H_2) = a_2^{2,\ 1}$, Theorem 4 implies $BR^1(\overline{s}_2) \subseteq BR^1(s_2)$.

If $A_1 = \emptyset$, then either $BR^1(\overline{s}_2) = BR^1(s_2)$ or $BR^1(s_2) = S_1^\Gamma(\overline{H}^{\underline{2}})$. Then the corresponding payoffs are $r_1^{2,\ 4}(> r_1^{2,\ 2}), r_1^{2,\ 3}(\geq r_1^{2,\ 1}) > r_1^{4,\ 1},\ r_1^{4,\ 2},\ r_1^{4,\ 3},\ r_1^{4,\ 4}$.

If $A_1 = S_1^\Gamma(\overline{H}^{\underline{2},\ \underline{2}})$, then $BR^1(\overline{s}_2) = BR^1(s_2)$. The corresponding payoffs are $r_1^{2,\ 1}(= r_1^{2,\ 3}), r_1^{2,\ 2}(= r_1^{2,\ 4}) > r_1^{4,\ 1},\ r_1^{4,\ 2},\ r_1^{4,\ 3},\ r_1^{4,\ 4}$

Under these payoff restrictions, the two-stage game satisfies strategic complementarities.

ii. Let $inf_{s_1 \in BR^1(\underline{s}_2)} s_1(h^0) = a_1^{1,\ 1}$ and $sup_{s_1 \in BR^1(\overline{s}_2)} s_1(h^0) = a_1^{1,\ 2}$.

(i). Suppose for all $s_2 \in S_2^\Gamma$ such that $s_2(h_2^0) = a_2^{1,\ 2}$, $BR^1(s_2) \subseteq \{s_1 \in S_1^\Gamma |\ s_1(h^0) =$

$a_1^{1,\ 2}\}$.

Under this assumption, consider all $s_2 \in S_2^{\Gamma}$ such that $s_2(h_2^0) = a_2^{1,\ 2}$.

Theorem 1 implies $BR^1(\overline{s}_2) = S_1^{\Gamma}(\overline{H}^{\overline{2},\ \overline{2}}) \cup A_1$ with $A_1 \in \{\emptyset,\ S_1^{\Gamma}(\overline{H}^{\overline{2},\ \underline{2}})\}$, that is, if $A_1 = \emptyset$, then $a_1^{4,\ 4}(> a_1^{4,\ 2}) > a_1^{2,\ 2},\ a_1^{2,\ 4}$, if $A_1 \neq \emptyset$, then $a_1^{4,\ 4}(= a_1^{4,\ 2}) > a_1^{2,\ 2},\ a_1^{2,\ 4}$.

For $s_2 \in S_2^{\Gamma}(\overline{H}^{\overline{2}})$, $s_2(\overline{H}^{\underline{2}} \cap H_2) = a_2^{2,\ 1}$ and $s_2(\overline{H}^{\overline{2}} \cap H_2) = a_2^{2,\ 1}$.

If $A_1 = \emptyset$, then either (1). $BR^1(s_2) = BR^1(\overline{s}_2)$, that is, $r_1^{4,\ 3}(> r_1^{4,\ 1}) >\ r_1^{2,\ 1},\ r_1^{2,\ 3}$, or (2). $BR^1(s_2) = BR^1(\overline{s}_2) \cup S_1^{\Gamma}(\overline{H}^{\overline{2},\ \underline{1}})$, that is, $r_1^{4,\ 3}(= r_1^{4,\ 1}) >\ r_1^{2,\ 1},\ r_1^{2,\ 3}$

If $A_1 = S_1^{\Gamma}(\overline{H}^{\overline{2},\ \underline{2}})$, then $BR^1(s_2) = BR^1(\overline{s}_2)$, that is, $r_1^{4,\ 3}(= r_1^{4,\ 1}) >\ r_1^{2,\ 1},\ r_1^{2,\ 3}$.

For $s_2 \in S_2^{\Gamma}(\overline{H}^{\overline{2}})$, $s_2(\overline{H}^{\underline{2}} \cap H_2) = a_2^{2,\ 2}$ and $s_2(\overline{H}^{\overline{2}} \cap H_2) = a_2^{2,\ 1}$,

If $A_1 = \emptyset$ and (1), then $BR^1(s_2) = BR^1(\overline{s}_2)$, that is, $r_1^{4,\ 3}(> r_1^{4,\ 1}) >\ r_1^{2,\ 2},\ r_1^{2,\ 4}$;

If $A_1 = \emptyset$ and (2), then $BR^1(s_2) = BR^1(\overline{s}_2) \cup S_1^{\Gamma}(\overline{H}^{\overline{2},\ \underline{1}})$, that is, $r_1^{4,\ 3}(= r_1^{4,\ 1}) > r_1^{2,\ 2},\ r_1^{2,\ 4}$,

If $A_1 = S_1^{\Gamma}(\overline{H}^{\overline{2},\ \underline{2}})$, then $BR^1(s_2) = BR^1(\overline{s}_2)$, that is, $r_1^{4,\ 3}(= r_1^{4,\ 1}) >\ r_1^{2,\ 2},\ r_1^{2,\ 4}$.

For $s_2 \in S_2^{\Gamma}(\overline{H}^{\overline{2}})$, $s_2(\overline{H}^{\underline{2}} \cap H_2) = a_2^{2,\ 1}$ and $s_2(\overline{H}^{\overline{2}} \cap H_2) = a_2^{2,\ 2}$,

$BR^1(s_2) = BR^1(\overline{s}_2)$, that is, if $A_1 = \emptyset$, then $r_1^{4,\ 4}(> r_1^{4,\ 2}) >\ r_1^{2,\ 1},\ r_1^{2,\ 3}$, if $A_1 \neq \emptyset$, then $r_1^{4,\ 4}(= r_1^{4,\ 2}) >\ r_1^{2,\ 1},\ r_1^{2,\ 3}$

Now consider all $s_2 \in S_2^{\Gamma}$ such that $s_2(h_2^0) = a_2^{1,\ 1}$,

Theorem 1 implies $BR^1(\underline{s}_2) = S_1^{\Gamma}(\overline{H}^{\underline{1},\ \underline{1}}) \cup A_1 \cup B$.

For $s_2 \in S_2^{\Gamma}(\overline{H}^{\underline{1}})$, $s_2(\overline{H}^{\underline{1}} \cap H_2) = a_2^{2,\ 2}$ and $s_2(\overline{H}^{\overline{1}} \cap H_2) = a_2^{2,\ 2}$. As implied by Theorem 3, either $BR^1(\underline{s}_2) \subseteq BR^1(s_2)$ or $BR^1(s_2) = B$.

If $A_1(\underline{s}_2) = \emptyset$ and $B(\underline{s}_2) = \emptyset$, then either (1). $BR^1(s_2) = BR^1(\underline{s}_2)$, that is, $r_1^{1,\,2}(> r_1^{1,\,4})) > r_1^{3,\,2},\ r_1^{3,\,4}$, or (2). $BR^1(s_2) = BR^1(\underline{s}_2) \cup S_2^{\Gamma}(\overline{H}^{\underline{1},\,\overline{1}})$, that is, $r_1^{1,\,2}(= r_1^{1,\,4}) > r_1^{3,\,2},\ r_1^{3,\,4}$, or (3). $BR^1(s_2) = BR^1(\underline{s}_2) \cup B$, that is, $r_1^{1,\,2}(> r_1^{1,\,4}) = r_1^{3,\,2} = r_1^{3,\,4}$, or (4). $BR^1(s_2) = BR^1(\underline{s}_2) \cup S_2^{\Gamma}(\overline{H}^{\underline{1},\,\overline{1}}) \cup B$, that is, $r_1^{1,\,2}(= r_1^{1,\,4}) = r_1^{3,\,2} = r_1^{3,\,4}$, or (5). $BR^1(s_2) = B$, that is, $r_1^{3,\,2} = r_1^{3,\,4} > r_1^{1,\,2},\ r_1^{1,\,4}$.

If $A_1(\underline{s}_2) = S_1^{\Gamma}(\overline{H}^{\underline{1},\,\overline{1}})$ and $B(\underline{s}_2) = \emptyset$, then either (1). $BR^1(s_2) = BR^1(\underline{s}_2)$, that is, $r_1^{1,\,2}(= r_1^{1,\,4})) > r_1^{3,\,2},\ r_1^{3,\,4}$, or (2). $BR^1(s_2) = BR^1(\underline{s}_2) \cup B$, that is, $r_1^{1,\,2}(= r_1^{1,\,4}) = r_1^{3,\,2} = r_1^{3,\,4}$, or (3). $BR^1(s_2) = B$, that is, $r_1^{3,\,2} = r_1^{3,\,4} > r_1^{1,\,2},\ r_1^{1,\,4}$.

If $A_1(\underline{s}_2) = \emptyset$ and $B(\underline{s}_2) = S_1^{\Gamma}(\overline{H}^{\overline{1}})$, then either (1) $BR^1(s_2) = BR^1(\underline{s}_2)$, that is, $r_1^{1,\,2}(> r_1^{1,\,4}) = r_1^{3,\,2} = r_1^{3,\,4}$, or (2) $BR^1(s_2) = B$, that is, $r_1^{3,\,2} = r_1^{3,\,4} > r_1^{1,\,2},\ r_1^{1,\,4}$.

If $A_1(\underline{s}_2) = S_1^{\Gamma}(\overline{H}^{\underline{1},\,\overline{1}})$ and $B(\underline{s}_2) = S_1^{\Gamma}(\overline{H}^{\overline{1}})$, then either (1) $BR^1(s_2) = BR^1(\underline{s}_2)$, that is, $r_1^{1,\,2} = r_1^{1,\,4} = r_1^{3,\,2} = r_1^{3,\,4}$, or (2) $BR^1(s_2) = B$, that is, $r_1^{3,\,2} = r_1^{3,\,4} > r_1^{1,\,2},\ r_1^{1,\,4}$

For $s_2 \in S_2^{\Gamma}(\overline{H}^{\underline{1},\,\underline{1}})$ such that $s_2(\overline{H}^{\overline{1}} \cap H_2) = a_2^{2,\,2}$, either $BR^1(\underline{s}_2) \cup B = BR^1(s_2)$ or $BR^1(s_2) = B$.

If $A_1 = \emptyset$ and $B = \emptyset$, under (1), $BR^1(s_2) = BR^1(\underline{s}_2)$, that is, $r_1^{1,\,1}(> r_1^{1,\,3}) > r_1^{3,\,2},\ r_1^{3,\,4}$; under (2). $BR^1(s_2) = BR^1(\underline{s}_2)$, that is, $r_1^{1,\,1}(> r_1^{1,\,3}) > r_1^{3,\,2},\ r_1^{3,\,4}$; under (3) and (4). then either $BR^1(s_2) = BR^1(\underline{s}_2)$, that is, $r_1^{1,\,1}(> r_1^{1,\,3}) > r_1^{3,\,2},\ r_1^{3,\,4}$, or $BR^1(s_2) = BR^1(\underline{s}_2) \cup B$, that is, $r_1^{1,\,1}(> r_1^{1,\,3}) = r_1^{3,\,2} = r_1^{3,\,4}$; under (5). then either $BR^1(s_2) = BR^1(\underline{s}_2)$, that is, $r_1^{1,\,1}(> r_1^{1,\,3}) > r_1^{3,\,2},\ r_1^{3,\,4}$, or $BR^1(s_2) = BR^1(\underline{s}_2) \cup B$, that is, $r_1^{1,\,1}(> r_1^{1,\,3}) = r_1^{3,\,2} = r_1^{3,\,4}$ or $BR^1(s_2) = B$, that is, $r_1^{3,\,2} = r_1^{3,\,4} > r_1^{1,\,1}(> r_1^{1,\,3})$.

If $A_1 = S_1^{\Gamma}(\overline{H}^{\underline{1},\,\overline{1}})$ and $B = \emptyset$, under (1), $BR^1(s_2) = BR^1(\underline{s}_2)$, that is, $r_1^{1,\,1}(= r_1^{1,\,3}) > r_1^{3,\,2},\ r_1^{3,\,4}$; under (2). $BR^1(s_2) = BR^1(\underline{s}_2)$, that is, $r_1^{1,\,1}(= r_1^{1,\,3}) > r_1^{3,\,2},\ r_1^{3,\,4}$ or

$BR^1(s_2) = BR^1(\underline{s}_2) \cup B$, that is, $r_1^{1,\ 1}(= r_1^{1,\ 3}) = r_1^{3,\ 2} = r_1^{3,\ 4}$; under (3). then either $BR^1(s_2) = BR^1(\underline{s}_2)$, that is, $r_1^{1,\ 1}(= r_1^{1,\ 3}) > r_1^{3,\ 2},\ r_1^{3,\ 4}$, or $BR^1(s_2) = BR^1(\underline{s}_2) \cup B$, that is, $r_1^{1,\ 1}(= r_1^{1,\ 3}) = r_1^{3,\ 2} = r_1^{3,\ 4}$ or $BR^1(s_2) = B$, that is, $r_1^{3,\ 2} = r_1^{3,\ 4} > r_1^{1,\ 1}(> r_1^{1,\ 3})$.

If $A_1 = \emptyset$ and $B = S_1^{\Gamma}(\overline{H}^{\bar{1}})$, under (1), $BR^1(s_2) = BR^1(\underline{s}_2)$, that is, $r_1^{1,\ 1}(> r_1^{1,\ 3}) = r_1^{3,\ 2} = r_1^{3,\ 4} = r_1^{3,\ 1} = r_1^{3,\ 3}$; under (2), either $BR^1(s_2) = BR^1(\underline{s}_2)$, that is, $r_1^{1,\ 1}(> r_1^{1,\ 3}) = r_1^{3,\ 2} = r_1^{3,\ 4}$ or $BR^1(s_2) = B$, that is, $r_1^{3,\ 2} = r_1^{3,\ 4} > r_1^{1,\ 1},\ r_1^{1,\ 3}$.

If $A_1 = S_1^{\Gamma}(\overline{H}^{\underline{1},\ \bar{1}})$ and $B = S_1^{\Gamma}(\overline{H}^{\bar{1}})$, under (1), $BR^1(s_2) = BR^1(\underline{s}_2)$, that is, $r_1^{1,\ 1}(= r_1^{1,\ 3}) = r_1^{3,\ 2} = r_1^{3,\ 4}$; under (2), either $BR^1(s_2) = BR^1(\underline{s}_2)$, that is, $r_1^{1,\ 1}(= r_1^{1,\ 3}) = r_1^{3,\ 2} = r_1^{3,\ 4}$ or $BR^1(s_2) = B$, that is, $r_1^{3,\ 2} = r_1^{3,\ 4} > r_1^{1,\ 1},\ r_1^{1,\ 3}$.

For $s_2 \in S_2^{\Gamma}(\overline{H}^{\underline{1}})$, $s_2(\overline{H}^{\underline{1}} \cap H_2) = a_2^{2,\ 2}$ and $s_2(\overline{H}^{\bar{1}} \cap H_2) = a_2^{2,\ 1}$.

If $A_1 = \emptyset$ and $B = \emptyset$, under (1), $BR^1(s_2) = BR^1(\underline{s}_2)$, that is, $r_1^{1,\ 2}(> r_1^{1,\ 4}) > r_1^{3,\ 1},\ r_1^{3,\ 3}$; under (2). $BR^1(s_2) = BR^1(\underline{s}_2) \cup S_1^{\Gamma}(\overline{H}^{\underline{1},\ \bar{2}})$, that is, $r_1^{1,\ 2}(= r_1^{1,\ 4}) > r_1^{3,\ 1},\ r_1^{3,\ 3}$; under (3) either $BR^1(s_2) = BR^1(\underline{s}_2)$, that is, $r_1^{1,\ 2}(> r_1^{1,\ 4}) > r_1^{3,\ 1},\ r_1^{3,\ 3}$, or $BR^1(s_2) = BR^1(\underline{s}_2) \cup B$, that is, $r_1^{1,\ 2}(> r_1^{1,\ 4}) = r_1^{3,\ 1} = r_1^{3,\ 3}$; under (4). either $BR^1(s_2) = BR^1(\underline{s}_2) \cup S_1^{\Gamma}(\overline{H}^{\underline{1},\ \bar{2}})$, that is, $r_1^{1,\ 2}(= r_1^{1,\ 4}) > r_1^{3,\ 1},\ r_1^{3,\ 3}$, or $BR^1(s_2) = BR^1(\underline{s}_2) \cup S_1^{\Gamma}(\overline{H}^{\underline{1},\ \bar{2}}) \cup B$, that is, $r_1^{1,\ 2}(= r_1^{1,\ 4}) = r_1^{3,\ 1} = r_1^{3,\ 3}$; under (5). either $BR^1(s_2) = BR^1(\underline{s}_2)$, that is, $r_1^{1,\ 2}(> r_1^{1,\ 4}) > r_1^{3,\ 1},\ r_1^{3,\ 3}$, or $BR^1(s_2) = BR^1(\underline{s}_2) \cup S_1^{\Gamma}(\overline{H}^{\underline{1},\ \bar{2}})$, that is, $r_1^{1,\ 2}(= r_1^{1,\ 4}) > r_1^{3,\ 1},\ r_1^{3,\ 3}$, or $BR^1(s_2) = BR^1(\underline{s}_2) \cup B$, that is, $r_1^{1,\ 2}(> r_1^{1,\ 4}) = r_1^{3,\ 1} = r_1^{3,\ 3}$, or $BR^1(s_2) = BR^1(\underline{s}_2) \cup S_1^{\Gamma}(\overline{H}^{\underline{1},\ \bar{2}}) \cup B$, that is, $r_1^{1,\ 2}(= r_1^{1,\ 4}) = r_1^{3,\ 1} = r_1^{3,\ 3}$, or $BR^1(s_2) = B$, that is, $r_1^{3,\ 1} = r_1^{3,\ 3} > r_1^{1,\ 2},\ r_1^{1,\ 4}$.

If $A_1 = S_1^{\Gamma}(\overline{H}^{\underline{1},\ \bar{1}})$ and $B = \emptyset$, under (1). $BR^1(s_2) = BR^1(\underline{s}_2)$, that is, $r_1^{1,\ 2}(= r_1^{1,\ 4}) > r_1^{3,\ 1},\ r_1^{3,\ 3}$; under (2). either $BR^1(s_2) = BR^1(\underline{s}_2)$, that is, $r_1^{1,\ 2}(= r_1^{1,\ 4}) > r_1^{3,\ 1},\ r_1^{3,\ 3}$, or $BR^1(s_2) = BR^1(\underline{s}_2) \cup B$, that is, $r_1^{1,\ 2}(= r_1^{1,\ 4}) = r_1^{3,\ 1} = r_1^{3,\ 3}$; under (3). either $BR^1(s_2) = BR^1(\underline{s}_2)$, that is, $r_1^{1,\ 2}(= r_1^{1,\ 4}) > r_1^{3,\ 1},\ r_1^{3,\ 3}$, or $BR^1(s_2) = BR^1(\underline{s}_2) \cup B$, that is, $r_1^{1,\ 2}(= r_1^{1,\ 4}) = r_1^{3,\ 1} = r_1^{3,\ 3}$ or $BR^1(s_2) = B$, that is, $r_1^{3,\ 1} = r_1^{3,\ 3} > r_1^{1,\ 2},\ r_1^{1,\ 4}$.

If $A_1 = \emptyset$ and $B = S_1^{\Gamma}(\overline{H}^{\bar{1}})$, then under (1). $BR^1(s_2) = BR^1(\underline{s}_2)$, that is, $r_1^{1,\ 2}(> r_1^{1,\ 4}) = r_1^{3,\ 1} = r_1^{3,\ 3}$; under (2). either $BR^1(s_2) = BR^1(\underline{s}_2)$, that is, $r_1^{1,\ 2}(> r_1^{1,\ 4}) = r_1^{3,\ 1} = r_1^{3,\ 3}$ or $BR^1(s_2) = B$, that is, $r_1^{3,\ 1} = r_1^{3,\ 3} > r_1^{1,\ 2},\ r_1^{1,\ 4}$.

If $A_1 = S_1^{\Gamma}(\overline{H}^{\underline{1},\ \bar{1}})$ and $B = S_1^{\Gamma}(\overline{H}^{\bar{1}})$, then under (1). $BR^1(s_2) = BR^1(\underline{s}_2)$, that is, $r_1^{1,\ 2}(= r_1^{1,\ 4}) = r_1^{3,\ 1} = r_1^{3,\ 3}$; under (2). either $BR^1(s_2) = BR^1(\underline{s}_2)$, that is, $r_1^{1,\ 2}(= r_1^{1,\ 4}) = r_1^{3,\ 1} = r_1^{3,\ 3}$ or $BR^1(s_2) = B$, that is, $r_1^{3,\ 1} = r_1^{3,\ 3} > r_1^{1,\ 2},\ r_1^{1,\ 4}$

Thus under those conditions, the game yields strategic complementarities.

(ii). Suppose there exists $s_2 \in S_2^{\Gamma}$, $s_2(h_2^0) = a_2^{1,\ 2}$ such that $BR^1(s_2) \cap \{s_1 \in S_1^{\Gamma} |\ s_1(h^0) = a_1^{1,\ 1}\} \neq \emptyset$. Then for all $s_2 \in S_2^{\Gamma}$ such that $s_2(h_2^0) = a_2^{1,\ 1}$, $BR^1(s_2) \cap \{s_1 \in S_1^{\Gamma} |\ s_1(h^0) = a_1^{1,\ 1}\} \neq \emptyset$.

For all $s_2 \in S_2^{\Gamma}$, $s_2(h_2^0) = a_2^{1,\ 2}$, the conditions to generate strategic complementarities is similar to the analysis of s_2 with $s_2(h_2^0) = a_1^{1,\ 1}$ in (i).

For s_2 such that $s_2(h_2^0) = a_2^{1,\ 1}$,

Theorem 1 implies $BR^1(\underline{s}_2) = S_1^{\Gamma}(\overline{H}^{\underline{1},\ \underline{1}}) \cup A_1 \cup B$.

For $s_2 \in S_2^{\Gamma}(\overline{H}^{\underline{1}})$, $s_2(\overline{H}^{\underline{1}} \cap H_2) = a_2^{2,\ 2}$ and $s_2(\overline{H}^{\bar{1}} \cap H_2) = a_2^{2,\ 2}$. $BR^1(\underline{s}_2) \subseteq BR^1(s_2)$.

If $A_1(\underline{s}_2) = \emptyset$ and $B(\underline{s}_2) = \emptyset$, then either (1). $BR^1(s_2) = BR^1(\underline{s}_2)$, that is, $r_1^{1,\ 2}(> r_1^{1,\ 4})) > r_1^{3,\ 2},\ r_1^{3,\ 4}$, or (2). $BR^1(s_2) = BR^1(\underline{s}_2) \cup S_2^{\Gamma}(\overline{H}^{\underline{1},\ \bar{1}})$, that is, $r_1^{1,\ 2}(= r_1^{1,\ 4}) > r_1^{3,\ 2},\ r_1^{3,\ 4}$, or (3). $BR^1(s_2) = BR^1(\underline{s}_2) \cup B$, that is, $r_1^{1,\ 2}(> r_1^{1,\ 4}) = r_1^{3,\ 2} = r_1^{3,\ 4}$, or (4).

$BR^1(s_2) = BR^1(\underline{s}_2) \cup S_2^\Gamma(\overline{H}^{\underline{1},\ \overline{1}}) \cup B$, that is, $r_1^{1,\ 2}(= r_1^{1,\ 4}) = r_1^{3,\ 2} = r_1^{3,\ 4}$.

If $A_1(\underline{s}_2) = S_1^\Gamma(\overline{H}^{\underline{1},\ \overline{1}})$ and $B(\underline{s}_2) = \emptyset$, then either (1). $BR^1(s_2) = BR^1(\underline{s}_2)$, that is, $r_1^{1,\ 2}(= r_1^{1,\ 4})) > r_1^{3,\ 2},\ r_1^{3,\ 4}$, or (2). $BR^1(s_2) = BR^1(\underline{s}_2) \cup B$, that is, $r_1^{1,\ 2}(= r_1^{1,\ 4}) = r_1^{3,\ 2} = r_1^{3,\ 4}$.

If $A_1(\underline{s}_2) = \emptyset$ and $B(\underline{s}_2) = S_1^\Gamma(\overline{H}^{\overline{1}})$, then $BR^1(s_2) = BR^1(\underline{s}_2)$, that is, $r_1^{1,\ 2}(> r_1^{1,\ 4}) = r_1^{3,\ 2} = r_1^{3,\ 4}$.

If $A_1(\underline{s}_2) = S_1^\Gamma(\overline{H}^{\underline{1},\ \overline{1}})$ and $B(\underline{s}_2) = S_1^\Gamma(\overline{H}^{\overline{1}})$, then either (1) $BR^1(s_2) = BR^1(\underline{s}_2)$, that is, $r_1^{1,\ 2} = r_1^{1,\ 4} = r_1^{3,\ 2} = r_1^{3,\ 4}$.

For $s_2 \in S_2^\Gamma(\overline{H}^{\underline{1},\ \underline{1}})$ such that $s_2(\overline{H}^{\overline{1}} \cap H_2) = a_2^{2,\ 2}$, either $BR^1(\underline{s}_2) \cup B = BR^1(s_2)$.

If $A_1 = \emptyset$ and $B = \emptyset$, under (1), $BR^1(s_2) = BR^1(\underline{s}_2)$, that is, $r_1^{1,\ 1}(> r_1^{1,\ 3}) > r_1^{3,\ 2},\ r_1^{3,\ 4}$; under (2). $BR^1(s_2) = BR^1(\underline{s}_2)$, that is, $r_1^{1,\ 1}(> r_1^{1,\ 3}) > r_1^{3,\ 2},\ r_1^{3,\ 4}$; under (3) and (4). then either $BR^1(s_2) = BR^1(\underline{s}_2)$, that is, $r_1^{1,\ 1}(> r_1^{1,\ 3}) > r_1^{3,\ 2},\ r_1^{3,\ 4}$, or $BR^1(s_2) = BR^1(\underline{s}_2) \cup B$, that is, $r_1^{1,\ 1}(> r_1^{1,\ 3}) = r_1^{3,\ 2} = r_1^{3,\ 4}$.

If $A_1 = S_1^\Gamma(\overline{H}^{\underline{1},\ \overline{1}})$ and $B = \emptyset$, under (1), $BR^1(s_2) = BR^1(\underline{s}_2)$, that is, $r_1^{1,\ 1}(= r_1^{1,\ 3}) > r_1^{3,\ 2},\ r_1^{3,\ 4}$; under (2). $BR^1(s_2) = BR^1(\underline{s}_2)$, that is, $r_1^{1,\ 1}(= r_1^{1,\ 3}) > r_1^{3,\ 2},\ r_1^{3,\ 4}$ or $BR^1(s_2) = BR^1(\underline{s}_2) \cup B$, that is, $r_1^{1,\ 1}(= r_1^{1,\ 3}) = r_1^{3,\ 2} = r_1^{3,\ 4}$.

If $A_1 = \emptyset$ and $B = S_1^\Gamma(\overline{H}^{\overline{1}})$, under (1), $BR^1(s_2) = BR^1(\underline{s}_2)$, that is, $r_1^{1,\ 1}(> r_1^{1,\ 3}) = r_1^{3,\ 2} = r_1^{3,\ 4} = r_1^{3,\ 1} = r_1^{3,\ 3}$.

If $A_1 = S_1^\Gamma(\overline{H}^{\underline{1},\ \overline{1}})$ and $B = S_1^\Gamma(\overline{H}^{\overline{1}})$, under (1), $BR^1(s_2) = BR^1(\underline{s}_2)$, that is, $r_1^{1,\ 1}(= r_1^{1,\ 3}) = r_1^{3,\ 2} = r_1^{3,\ 4}$.

For $s_2 \in S_2^\Gamma(\overline{H}^{\underline{1}})$, $s_2(\overline{H}^{\underline{1}} \cap H_2) = a_2^{2,\ 2}$ and $s_2(\overline{H}^{\overline{1}} \cap H_2) = a_2^{2,\ 1}$.

If $A_1 = \emptyset$ and $B = \emptyset$, under (1), $BR^1(s_2) = BR^1(\underline{s}_2)$, that is, $r_1^{1,\ 2}(> r_1^{1,\ 4}) > r_1^{3,\ 1},\ r_1^{3,\ 3}$; under (2). $BR^1(s_2) = BR^1(\underline{s}_2) \cup S_1^\Gamma(\overline{H}^{1,\ \bar{2}})$, that is, $r_1^{1,\ 2}(= r_1^{1,\ 4}) > r_1^{3,\ 1},\ r_1^{3,\ 3}$; under (3) either $BR^1(s_2) = BR^1(\underline{s}_2)$, that is, $r_1^{1,\ 2}(> r_1^{1,\ 4}) > r_1^{3,\ 1},\ r_1^{3,\ 3}$, or $BR^1(s_2) = BR^1(\underline{s}_2) \cup B$, that is, $r_1^{1,\ 2}(> r_1^{1,\ 4}) = r_1^{3,\ 1} = r_1^{3,\ 3}$; under (4). either $BR^1(s_2) = BR^1(\underline{s}_2) \cup S_1^\Gamma(\overline{H}^{1,\ \bar{2}})$, that is, $r_1^{1,\ 2}(= r_1^{1,\ 4}) > r_1^{3,\ 1},\ r_1^{3,\ 3}$, or $BR^1(s_2) = BR^1(\underline{s}_2) \cup S_1^\Gamma(\overline{H}^{1,\ \bar{2}}) \cup B$, that is, $r_1^{1,\ 2}(= r_1^{1,\ 4}) = r_1^{3,\ 1} = r_1^{3,\ 3}$.

If $A_1 = S_1^\Gamma(\overline{H}^{1,\ \bar{1}})$ and $B = \emptyset$, under (1). $BR^1(s_2) = BR^1(\underline{s}_2)$, that is, $r_1^{1,\ 2}(= r_1^{1,\ 4}) > r_1^{3,\ 1},\ r_1^{3,\ 3}$; under (2). either $BR^1(s_2) = BR^1(\underline{s}_2)$, that is, $r_1^{1,\ 2}(= r_1^{1,\ 4}) > r_1^{3,\ 1},\ r_1^{3,\ 3}$, or $BR^1(s_2) = BR^1(\underline{s}_2) \cup B$, that is, $r_1^{1,\ 2}(= r_1^{1,\ 4}) = r_1^{3,\ 1} = r_1^{3,\ 3}$.

If $A_1 = \emptyset$ and $B = S_1^\Gamma(\overline{H}^{\bar{1}})$, then under (1). $BR^1(s_2) = BR^1(\underline{s}_2)$, that is, $r_1^{1,\ 2}(> r_1^{1,\ 4}) = r_1^{3,\ 1} = r_1^{3,\ 3}$.

If $A_1 = S_1^\Gamma(\overline{H}^{1,\ \bar{1}})$ and $B = S_1^\Gamma(\overline{H}^{\bar{1}})$, then under (1). $BR^1(s_2) = BR^1(\underline{s}_2)$, that is, $r_1^{1,\ 2}(= r_1^{1,\ 4}) = r_1^{3,\ 1} = r_1^{3,\ 3}$.

4.4 A Special Case

Assumption 1. *For every $s_2 \in \{s_2 \in S_2^\Gamma |\ s_2(h_2^0) = a_2^{1,\ m}\}$, there exists $a_1^{1,\ n} \in A_1^1$ such that $BR^1(s_2) \subseteq \{s_1 \in S_1^\Gamma |\ s_1(h^0) = a_1^{1,\ n}\}$.*

Under this assumption, for arbitrary $s_2 \in S_2^\Gamma$ with $s_2(h^0) = a_2^{1,\ 1}$, there exists $a_1^{1,\ i} \in A_1^1$ such that $BR^1(s_2) \subset \{s_1 \in S_1^\Gamma |\ s_1(h^0) = a_1^{1,\ i}\}$. Thus for $\underline{s}_2$, $a_1^{1,\ i} = inf_{s_1 \in BR^1(\underline{s}_2)} s_1(h^0)$. For arbitrary $s_2 \in S_2^\Gamma$ with $s_2(h^0) = a_2^{1,\ n^1}$, there exists $a_1^{1,\ j} \in A_1^1$ such that $BR^1(s_2) \subset \{s_1 \in S_1^\Gamma |\ s_1(h^0) = a_1^{1,\ j}\}$. Thus for $\overline{s}_2$, $a_1^{1,\ j} = sup_{s_1 \in BR^1(\overline{s}_2)} s_1(h^0)$. Thus the results in Lemma 2 applies here.

Lemma 16. *For arbitrary* $\hat{s}_2,\ \tilde{s}_2 \in \{s_2 \in S_2^\Gamma |\ s_2(h_2^0) \in A_2^1 \backslash \{a_2^{1,\ 1},\ a_2^{1,\ n^1}\}\}$ *such that* $\hat{s}_2(h_2^0) \prec \tilde{s}_2(h_2^0)$, $BR^1(\hat{s}_2) \sqsubseteq BR^1(\tilde{s}_2)$.

Proof. Lemma 6 and Assumption 1 indicate that there exists $a_1^{1,\ \hat{m}},\ a_1^{1,\ \tilde{m}} \in A_1^1$ such that $BR^1(\hat{s}_2) = \{s_1 \in S_1^\Gamma |\ s_1(h^0) = a_1^{1,\ \hat{m}}\}$ and $BR^1(\tilde{s}_2) = \{s_1 \in S_1^\Gamma |\ s_1(h^0) = a_1^{1,\ \tilde{m}}\}$, both of them are complete lattices. Let $\hat{s}_2, \tilde{s}_2 \in \overline{S}_2$, since $\hat{s}_2(h_2^0) \prec \tilde{s}_2(h_2^0)$, $\hat{s}_2 \prec \tilde{s}_2$. Thus for arbitrary $\hat{s}_1 \in BR^1(\hat{s}_2)$ and $\tilde{s}_1 \in BR^1(\tilde{s}_2)$, strategic complementarities imply that $\hat{s}_1 \vee \tilde{s}_1 \in BR^1(\tilde{s}_2)$, in particular, $(\hat{s}_1 \vee \tilde{s}_1)(h^0) = a_1^{1,\ \hat{m}} \vee a_1^{1,\ \tilde{m}} = a_1^{1,\ \tilde{m}}$. Thus $\hat{m} \leq \tilde{m}$. Thus $BR^1(\hat{s}_2) \sqsubseteq BR^1(\tilde{s}_2)$.

Assumption 2. *Consider* $\underline{s}_2,\ \overline{s}_2 \in S_2^\Gamma$*:*

a. $BR^1(\underline{s}_2) = S_1^\Gamma(\overline{H}^{1(i),\ \underline{1}^2,...,\ \underline{1}^T})$

b. $BR^1(\overline{s}_2) = S_1^\Gamma(\overline{H}^{n^1(j),\ \overline{n}^2,...,\ \overline{n}^T})$.

Assumption 2(a) requires Subgame $\overline{H}^{1(i),\ \underline{1}^2,...,\ \underline{1}^{T-1}}$ to be reached on all the best response paths of $\underline{s}_2$ and $a_1^{T,\ 1}$ is assigned to be played in that subgame by player 1. 2(b) requires Subgame $\overline{H}^{n^1(j),\ \overline{n}^2,...,\ \overline{n}^{T-1}}$ to be reached on all the best response paths of $\overline{s}_2$ and $a_1^{T,\ n^T}$ is assigned to be played in that subgame by player 1.

Notice here, $S_1^\Gamma(\overline{H}^{1(i),\ \underline{1}^2,...,\ \underline{1}^T}) = \{s_1 \in S_1^\Gamma |\ s_1(h^0) = a_1^{1,\ i},\ s_1(\overline{H}^{1(i)} \cap H^1) = a_1^{2,\ 1},\ s_1(\overline{H}^{1(i),\ \underline{1}^2} \cap H^1) = a_1^{3,\ 1}, ...,\ s_1(\overline{H}^{1(i),\ \underline{1}^2,...,\ \underline{1}^{T-1}} \cap H_1) = a_1^{T,\ 1}\}$ and $S_1^\Gamma(\overline{H}^{n^1(j),\ \overline{n}^2,...,\ \overline{n}^T}) = \{s_1 \in S_1^\Gamma |\ s_1(h^0) = a_1^{1,\ j},\ s_1(\overline{H}^{n^1(j)} \cap H^1) = a_1^{2,\ n^2},\ s_1(\overline{H}^{n^1(j),\ \overline{n}^2} \cap H^1) = a_1^{3,\ n^3}, ...,\ s_1(\overline{H}^{n^1(j),\ \overline{n}^2,...,\ \overline{n}^{T-1}} \cap H_1) = a_1^{T,\ n^T}\}$.

Assumption 3. *For arbitrary* $\hat{t}$, $1 < \hat{t} < T$, *for* $\hat{s}_2$ *such that*

a. For $\hat{s}_2 = sup S_2^\Gamma(\overline{H}^{1(i),\ \underline{1}^2,...,\ \underline{1}^{\hat{t}-1}})$, $BR^1(\hat{s}_2) \subseteq S_1^\Gamma(\overline{H}^{1(i),\ \underline{1}^2,...,\ \underline{1}^{\hat{t}-1},\ \underline{n}^{\hat{t}}})$

b. For $\hat{s}_2 = inf S_2^\Gamma(\overline{H}^{n^1(j),\ \overline{n}^2,...,\ \overline{n}^{\hat{t}-1}})$, $BR^1(\hat{s}_2) \subseteq S_1^\Gamma(\overline{H}^{n^1(j),\ \overline{n}^2,...,\ \overline{n}^{\hat{t}-1},\ \overline{1}^{\hat{t}}})$

Lemma 17. *Consider the following:*

a. If $s_2 \in S_2^\Gamma(\overline{H}^{1(i),\ \underline{1}^2 \ldots\ \underline{1}^{\hat{t}-1}})$ *and* $s_2(\overline{H}^{1(i),\ \underline{1}^2 \ldots\ \underline{1}^{\hat{t}-1}} \cap H_2) = a_2^{\hat{t},\ m^{\hat{t}}} \in A_2^{\hat{t}} \backslash a_2^{\hat{t},\ 1}$*, then* $BR^1(s_2) = S_1^\Gamma(\overline{H}^{1(i),\ \underline{1}^2 \ldots\ \underline{1}^{\hat{t}-1},\ \underline{m}^{\hat{t}}})$.

b. If $s_2 \in S_2^\Gamma(\overline{H}^{n^1(j),\ \overline{n}^2 \ldots\ \overline{n}^{\hat{t}-1}})$ *and* $s_2(\overline{H}^{n^1(j),\ \overline{n}^2 \ldots\ \overline{n}^{\hat{t}-1}} \cap H_2) = a_2^{\hat{t},\ m^{\hat{t}}} \in A_2^{\hat{t}} \backslash a_2^{\hat{t},\ n^{\hat{t}}}$*, then* $BR^1(s_2) = S_1^\Gamma(\overline{H}^{n^1(j),\ \overline{n}^2 \ldots\ \overline{n}^{\hat{t}-1},\ \overline{m}^{\hat{t}}})$.

Proof. a. Form $\hat{s}_2 = supS_2^\Gamma(\overline{H}^{1(i),\ \underline{1}^2, \ldots,\ \underline{1}^{\hat{t}-1}})$, Theorem 1 indicates that $S_1^\Gamma(\overline{H}^{1(i),\ \underline{1}^2, \ldots,\ \underline{1}^{\hat{t}-1},\ \underline{n}^{\hat{t}}}) \subseteq BR^1(\hat{s}_2)$. Assumption 5 implies $S_1^\Gamma(\overline{H}^{1(i),\ \underline{1}^2, \ldots,\ \underline{1}^{\hat{t}-1},\ \underline{n}^{\hat{t}}}) = BR^1(\hat{s}_2)$.

As $s_2 \prec \hat{s}_2$, for arbitrary $s_1 \in BR^1(s_2)$ and $\hat{s}_1 \in BR^1(\hat{s}_2)$, $s_1 \vee \hat{s}_1 \in BR^1(\hat{s}_2)$. Let $s_1 \vee \hat{s}_1 = \tilde{s}_1$. As $s_1 \preceq \tilde{s}_1$, $s_1(h^0) \preceq \tilde{s}_1(h^0) = a_1^{1,\ i}$, $s_1(\overline{H}^{1(i)} \cap H_1) \preceq \tilde{s}_1(\overline{H}^{1(i)} \cap H_1) = a_1^{2,\ 1}, \ldots,\ s_1(\overline{H}^{1(i),\ \underline{1}^2 \ldots\ \underline{1}^{\hat{t}-1}} \cap H_1) = a_1^{\hat{t},\ 1}$. So $s_1 \in \{s_1 \in S_1^\Gamma |\ s_1(h^0) = a_1^{1,\ i},\ s_1(\overline{H}^{1(i)} \cap H_1) = a_1^{2,\ 1} \ldots\ s_1(\overline{H}^{1(i),\ \underline{1}^2 \ldots\ \underline{1}^{\hat{t}-1}} \cap H_1) = a_1^{\hat{t},\ 1}\}$. Thus $BR^1(s_2) \subseteq \{s_1 \in S_1^\Gamma |\ s_1(h^0) = a_1^{1,\ i},\ s_1(\overline{H}^{1(i)} \cap H_1) = a_1^{2,\ 1} \ldots\ s_1(\overline{H}^{1(i),\ \underline{1}^2 \ldots\ \underline{1}^{\hat{t}-1}} \cap H_1) = a_1^{\hat{t},\ 1}\}$. Thus Theorem 1 implies $BR^1(s_2) = S_1^\Gamma(\overline{H}^{1(i),\ \underline{1}^2 \ldots\ \underline{1}^{\hat{t}-1},\ \underline{m}^{\hat{t}}})$.

b. proved similarly as a. ∎

4.5 Conclusion

For a general multi-stage 2-player game to exhibits strategic complements, interesting structures on the corresponding best response sets arise.

For example, if player 2 chooses the lowest possible strategy in this game, that is, at each player 2's information set, the lowest action is selected. Once player 1's first stage choice is fixed, player 1's best response set must include those strategies that select the lowest action on each player 1's information sets reached on the path of play. Suppose a higher action is chosen at the information set reached on such path of play, a diverge from original path of play occurs. Strategic complements will require player 1 to be indifferent among all the actions on the information sets reached after such information set thus generating more paths of play.

Thus, in response to different classes of player 2's strategies identified by the action selected in the first stage decision node, to maintain strategic complements, player 1's corresponding best response sets must include all the strategies that can generate the paths of play identified in the paper once the conditions are met.

5 Reference

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