MANAGEMENT SCIENCE
Vol. 26, No. 4, April 1980
Printed in U.S.A.

ON COMMITTEE DECISION MAKING: A GAME THEORETICAL APPROACH*

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In this paper, we study the committee decision making process using game theory. By a committee, we mean any group of people who have to pick one option from a given set of alternatives. A well defined voting rule is specified by which the committee arrives at a decision. Each member has a preference relation on the set of all alternatives. A new solution concept called the one-core is introduced and studied. Intuitively, the one-core consists of all maximal (for the proposer) proposals which are undominated assuming that the player who makes the proposal does not cooperate in any effort to dominate the proposal. For games with non-emtpy cores, the one-core proposals are shown to be better than the core. For games with empty cores, the one-core proposals tend to be pessimistic, i.e., they indicate the security level of the players. This is because the stability requirements of the one-core are too strong for such games. A bargaining set modeled along the lines of the Aumann-Maschler bargaining set for characteristic function games is defined for committee games. Because of its relaxed stability requirements, the bargaining set indicates more reasonable proposals than the one-core. The existence of both the one-core and the bargaining set are studied and these concepts are compared with two other well known solution concepts—the core and the Condorcet solution.
(GAMES/GROUP DECISIONS; VOTING/COMMITTEES)

1. Introduction

In this paper, we study the committee decision making process using game theory. By a *committee* we will mean any finite group of persons who have to pick one option from a given set of outcomes. The members of the committee will be situated in one room. This assumption is important for our model. We are primarily concerned with small committees that arrive at a decision after lengthy deliberations. In this respect we differ fundamentally from the theory of elections where the decision makers (the voters) are numerous and spread out extensively.

A well defined rule is specified by which the committee will arrive at a decision. The rules are designed such that the decision of the committee will consist of a unique outcome. We do not restrict ourselves to the straight majority rule only.

The set of alternatives (outcomes) may be finite or infinite. We will assume that there are at least two alternatives. One of the alternatives is always a 'status quo' outcome and will be denoted by a_0. a_0 will be the decision of the committee if it cannot agree on any other outcome or if it specifically picks a_0 to be the final decision. No agenda is specified. Any member is allowed to suggest any alternative at any time for consideration by the committee.

Each member of the committee has a preference relation on the set of all outcomes. We will assume that each member's preference relation is a weak order. (All undefined terms here are defined formally in Section 2.) For notational convenience, we shall represent each member's preference relation by means of an ordinal utility function

*Accepted by Ambar G. Rao; received February 1979. This paper has been with the author 4 months for 2 revisions.

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0025-1909/80/2604/0387$01.25

defined on the set of all outcomes. In this sense, utility is assumed to be nontransferable and interpersonal comparison of utilities is assumed to be meaningless.

Each member of the committee is assumed to be a "rational player" in the sense of von Neumann and Morgenstern [18], i.e. roughly speaking, each member is attempting to maximize his utility. Furthermore, we assume that the members make decisions under conditions of complete information, i.e. each member is completely aware of his own and everyone else's preference relation.

In the next section, the committee decision making process is modeled as a game.[1] Due to the special nature of the game, distinct from games with side payments, games without side payments, games in partition function form, abstract games, etc., we call such a game, a committee game.[2] In subsequent sections, some solutions of the game are studied. In particular, a new solution concept called the one-core is introduced. Also the concept of a bargaining set first introduced by Aumann and Maschler [4] in the context of games with side payments is defined with appropriate modifications for committee games. Both these solution concepts are studied in relation to each other and in relation to two other well known solution concepts, the Condorcet solution and the core.

Like most of game theory, our models of the committee decision making process are primarily normative in nature. We feel that it is unlikely that our solution concepts will predict outcomes in real life or in laboratory conditions where the members have little or no experience in committee decision making. However, if the committee members have had extensive experience (in decision making) or if the subjects of an experiment in laboratory conditions are aware of these models, then it is quite likely (in our opinion) that these models will be predictive. This is the only true test of a normative model. Our primary concern is not so much to predict as it is to understand much better the committee decision making process.

2. The Committee Game

Let $N = \{1, 2, \ldots, n\}$ denote the set of *players* (*committee members*). Let X denote the set of all *outcomes* (*alternatives*). X may consist of a finite or infinite number of outcomes. If X is finite we will denote X by $\{a_0, a_1, \ldots, a_m\}$ where $m \geqslant 1$. Outcome a_0 will be referred to as the *status quo outcome*.[3] Nonempty subsets of N will be called *coalitions*. Let $2^N = \{R \subset N : R \neq \emptyset\}$ denote the set of all coalitions and 2^X denote the set of all subsets of X. The rules by which the committee members arrive at a decision is called the *characteristic function* $v : 2^N \to 2^X$ and v is assumed to satisfy the following conditions:

$$\text{for each } R_1, R_2 \in 2^N, \quad R_1 \supset R_2 \Rightarrow v(R_1) \supset v(R_2), \tag{2.1}$$

$$v(N) = X, \tag{2.2}$$

$$\text{for each } R_1, R_2 \in 2^N, \quad R_1 \cap R_2 = \emptyset, v(R_1) \neq \emptyset, v(R_2) \neq \emptyset \Rightarrow v(R_1) = v(R_2) \text{ and } |v(R_1)| = 1. \tag{2.3}$$

[1]This modeling of a committee process as a game is not new. See, for example, Wilson [23].

[2]The definition of a committee game (see §2) comes closest to the concept of a characteristic function game without side payments. However, due to the discrete nature of the committee game, we shall not attempt to model the committee decision making process in terms of games without side payments.

[3]The status quo outcome is not used to define the characteristic function and hence does not really belong to the definition of the committee game. However, it will be used later to define a simple committee game.

$v(R)$ denotes the subset of outcomes that coalition R can realize if the decision is unanimous in R. Conditions (2.1) and (2.2) are intuitively obvious. Condition (2.3) ensures that the committee decision consists of at most one outcome.

Let $u_i : X \to E^1$ denote the real-valued ordinal *utility* function of player i, $i = 1, \ldots, n$. Utility is assumed to be nontransferable and interpersonal comparison of utilities has no meaning. The utility function u_i of player i is a representation of player i's preference relation $>_i$ on the set of outcomes as follows. Let $a, b \in X$. We write $a >_i b$ (i.e., a is strictly preferred to b by player i) iff $u_i(a) > u_i(b)$; and write $\sim(a >_i b)$ or $a \leqslant_i b$ (i.e. a is not strictly preferred to b by player i) iff $u_i(a) \leqslant u_i(b)$. Clearly the binary relations $\{>_i\}_{i \in N}$ on X are asymmetric, transitive and negatively transitive[4] (i.e. a weak order).[5] $\Gamma = (N, X, v, \{>_i\}_{i \in N})$ is then called *n-person committee game*. The game is said to to *finite* if X is a finite set and *infinite* otherwise. A committee game is said to be *simple* if

$$\text{for each } R \in 2^N, \quad v(R) = \emptyset \text{ or } v(R) = \{a_0\} \text{ or } v(R) = X.$$

If $v(R) = X$, we say R is a *winning* coalition and if $v(R) = \emptyset$ or $\{a_0\}$, we say R is a *losing* coalition. In addition, if $v(R) = \{a_0\}$, coalition R is also said to be a *blocking* coalition. Let $\mathfrak{W}$ denote the set of all winning coalitions, $\mathfrak{L}$ the set of all losing coalitions and $\mathfrak{B}$, the set of all blocking coalitions. Note that condition (2.3) requires the committee game to be *proper*, i.e. in any partition of N into coalitions, at most one coalition is winning. Player i is said to be a *dictator* if $\{i\} \in \mathfrak{W}$. If $\bigcap_{R \in \mathfrak{W}} R \neq \emptyset$ then $i \in \bigcap_{R \in \mathfrak{W}} R$ is said to be a *veto* player. A simple committee game is said to be *strong* if $\mathfrak{B} = \emptyset$. A coalition R is a *minimal winning coalition* if it is a winning coalition and if every proper subset of R is a losing coalition. Let $\mathfrak{W}^m$ denote the set of all minimal winning coalitions. Player i is said to be a *dummy* if $i \notin \bigcup_{R \in \mathfrak{W}^m} R$. Dummies play no essential role in a game and for all practical purposes can be omitted from the game. See Shapley [21] for a detailed description of simple games.

The characteristic function v and the preference relations $\{>_i\}_{i \in N}$ induce a natural dominance relation[6] on the set of outcomes, X. Let $a, b \in X$ and $R \in 2^N$. We say *a dominates b via R* denoted by $a \operatorname{dom}_R b$ iff

$$a >_i b \quad \text{for each } i \in R \quad \text{and} \tag{2.4}$$

$$a \in v(R). \tag{2.5}$$

Clearly, for each $R \in 2^N$, dom_R is asymmetric and transitive. Let $a, b \in X$. We say *a dominates b* denoted by $a \operatorname{dom} b$ iff

$$\text{there exists } R \in 2^N \text{ such that } a \operatorname{dom}_R b. \tag{2.6}$$

The binary relation dom may not be transitive. However we have the following result.

PROPOSITION 2.1. *The binary relation dom is asymmetric for all committee games.*

[4]A binary relation $>_i$ on x is said to be negatively transitive if for each $x, y, z \in X$,

$$x \leqslant_i y \quad \text{and} \quad y \leqslant_i z \Rightarrow x \leqslant_i z.$$

[5]The utility functions $\{u_i\}_{i=i}^n$ are introduced solely to define the preference relations. Alternatively, we could have simply assumed that for each $i \in N$, player i has a preference relation $>_i$ on X that is a weak order.

[6]We mimic the definition of domination given by von Neumann and Morgenstern [18] in the context of games with side payments.

PROOF. Suppose not. Then there exists $a, b \in X$ such that a dom b and b dom a, i.e., there exists $R_1, R_2 \in 2^N$ such that a dom_{R_1} b and b dom_{R_2} a. Now since the preference relations $\{>_i\}_{i \in N}$ are asymmetric, we have $R_1 \cap R_2 = \emptyset$. Also we have $a \in v(R_1)$, $b \in v(R_2)$ which contradicts condition (2.3). Q.E.D.

Before we proceed to study the solutions of these games, we introduce some notation. For each $a \in X$, let $D(a) = \{x \in X : a \text{ dom } x\}$ and $U(a) = X - D(a)$. Also for each $B \in 2^X$, B nonempty, let $D(B) = \bigcup_{x \in B} D(x)$ and $U(B) = X - D(B)$.

3. The Condorcet Solution and the Core

Let $\Gamma = (N, X, v, \{>_i\}_{i \in N})$ be a committee game. A *Condorcet solution* of the game Γ is an outcome $\alpha \in X$ such that it dominates every other outcome in X, i.e., $\{\alpha\} = X - D(\alpha) = U(\alpha)$.

An obvious result is as follows.

PROPOSITION 3.1. *For all committee games, if a Condorcet solution exists, then it is unique.*

The proof follows from the asymmetric property of the binary relation dom (Proposition 2.1).

The Condorcet solution was first defined by Condorcet [8] and rediscovered independently by Dodgson [10] (cf. Black [7]).

The *core* $\mathcal{C}$ of a committee game is the set of all undominated outcomes. I.e.,

$$\mathcal{C} = X - D(X) = U(X)$$

The core was first studied explicitly by Gillies [12] and Shapley.

An obvious relation between the Condorcet solution and the core is as follows.

PROPOSITION 3.2. *Let Γ be a committee game such that the Condorcet solution α exists. Then the core of the game coincides with the Condorcet solution, i.e.,*

$$\mathcal{C} = \{\alpha\}.$$

The proof follows from the asymmetric property of the binary relation dom (Proposition 2.1).

The converse, obviously, is not always true. The Condorcet solution has the strongest stability requirement of all solution concepts in game theory. However, this is attained at the cost of existence. The Condorcet solution doesn't always exist. The core represents the next level of stability requirements. These are not as strong as in the case of the Condorcet solution but strong enough to qualify as a viable solution concept. Unfortnately, the core is not always nonempty. The study of games with empty cores has been the traditional subject of study in game theory and also in the social sciences. Many different solutions have been forwarded. We do not intend to list these here. The problem is sufficiently complex to ensure that no single solution concept will suffice for all kinds of games. In the next section, we present a new solution concept called the one-core which results from a small modification in the definition of the core. The modification is motivated by behavioral considerations.

4. The One-Core

Let Γ be a committee game. A *proposal* is a pair (i, x) such that $i \in N$ and $x \in X$. A proposal (i, x) represents a motion x introduced by player i. Let $P = N \times X$ denote the set of all proposals. Define

$$\hat{C}^{(i)} = \{(i,x) \in P : x \text{ is not dominated via any coalition } R \subset N - \{i\}\}. \tag{4.1}$$

$\hat{C}^{(i)}$ represents the set of proposals made by i that are undominated assuming player i's noncooperation in any effort to dominate his proposal.

Let

$$C^{(i)} = \{(i,x) \in \hat{C}^{(i)} : y \leqslant_i x \text{ for each } (i,y) \in \hat{C}^{(i)}\}. \tag{4.2}$$

Intuitively, $C^{(i)}$ represents the maximal (best) proposals in the set $\hat{C}^{(i)}$ for player i.

The *one-core*, $\mathcal{C}_1$, of the game Γ is then defined by

$$\mathcal{C}_1 = \bigcup_{i \in N} C^{(i)}. \tag{4.3}$$

Intuitively, the one-core consists of all (maximal) proposals which are undominated assuming that the player who makes the proposal does not cooperate in any effort to dominate the proposal. For obvious reasons, assuming all proposals in $\hat{C}^{(i)}$ to be equally stable, player i picks only the maximal ones.

We now proceed to study the properties of the one-core.

THEOREM 4.1. *Let Γ be a committee game such that the Condorcet solution α exists. Then the one-core is given by*

$$\mathcal{C}_1 = \{(1,\alpha),(2,\alpha),\ldots,(n,\alpha)\}.$$

PROOF. From Proposition (3.2), it follows that $(i,\alpha) \in \hat{C}^{(i)}$ for each $i \in N$. We claim $C^{(i)} \supset \{(i,\alpha)\}$ for each $i \in N$. Suppose not. Let $(i,x) \in \hat{C}^{(i)}$ such that $x \neq \alpha$ and $x >_i \alpha$. Since α is a Condorcet solution $\alpha \operatorname{dom}_R x$ for some $R \in 2^N$. If $i \in R$, then $\alpha >_i x$, a contradiction! If $i \notin R$ then $(i,x) \notin \hat{C}^{(i)}$, a contradiction again! Hence $C^{(i)} \supset \{(i,\alpha)\}$. Using exactly the same argument as before, we can show that $(i,x) \in C^{(i)}$, $x \neq \alpha$ leads to a contradiction. Hence $C^{(i)} = \{(i,\alpha)\}$. Q.E.D.

The result is a strong endorsement for the one-core when the Condorcet solution exists.

THEOREM 4.2. *Let Γ be a finite committee game such that the core $\mathcal{C}$ is nonempty. Then $C^{(i)} \neq \emptyset$ for each $i \in N$. Furthermore $(i,x) \in C^{(i)} \Rightarrow y \leqslant_i x$ for each $y \in \mathcal{C}$.*

PROOF. $\mathcal{C} \neq \emptyset \Rightarrow \hat{C}^{(i)} \neq \emptyset$ for each $i \in N$. Since X is a finite set, $P = N \times X$ is finite. Hence $C^{(i)} \neq \emptyset$. Thus $\mathcal{C}_1 \neq \emptyset$. The second assertion follows from the definition of $C^{(i)}$. Q.E.D.

The theorem states that each player by proposing his proposals from the one-core does as good (if not better) than any outcome in the core. The finiteness condition is merely academic and is assumed to ensure that the maximal proposal exists. We can assume a much weaker condition than finiteness of X, but we think our point is well made.

The following example illustrates the advantage of the one-core over the core.

EXAMPLE 4.1. Let $N = \{1,2,3\}$, $X = \{a_0,a_1,a_2,a_3,a_4\}$, v be given by $v(1) = v(23) = \{a_0\}$, $v(2) = v(3) = \emptyset$, $v(12) = v(13) = v(123) = X$, and the utility functions are given by

	u_1	u_2	u_3
a_0	1	3	1
a_1	3	0	2
a_2	4	1	0
a_3	2	2	3
a_4	0	4	4

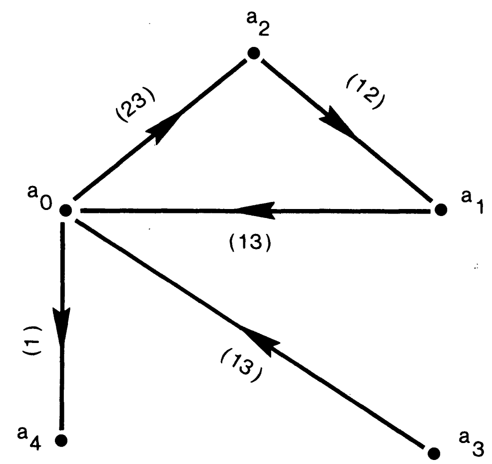

FIGURE 4.1. The Dominance Relations in Example 4.1.

Note that the decision rule is a straight majority rule with player 1 having veto power.

For this committee game, $\mathcal{C} = \{a_3\}$ and $\mathcal{C}_1 = \{(1, a_1), (2, a_3), (3, a_3)\}$. Note that player 1 by proposing his proposal $(1, a_1)$ can do better than the core. If after player 1 has introduced motion a_1, player 2 introduces motion a_2 and suggests a vote between the two, player 1 will vote for motion a_1 and thus prevent motion a_1 from being defeated. Player 1 is motivated to do so because if he votes for a_2 and a_1 is defeated, then it is possible that player 3 will suggest a_0 and player 1 is powerless to stop a_2 from being defeated by a_0 which puts him in a worse situation than he was in a_1. Thus player 1 benefits by anticipating more than one "step" ahead. We do not claim that a_1 will be the outcome of this game if played by rational players. As mentioned before, ours is a normative or prescriptive theory. The core prescribes a_3 as the solution of this game. This, in our opinion, is not very satisfactory. The one-core, on the other hand, prescribes that player 1 should propose a_1 to be the committee decision. Player 1 has as much 'right' to expect a_1 to be the committee decision as players 2 and 3 have to expect a_3 as the outcome. Although player 1 cannot enforce a_1 by himself, neither can player 2 and 3 enforce a_3 (remember that player 1 has veto power). The outcome that will actually be observed in an empirical setting will depend on which player makes the first proposal (the rules of the committee may decide this), the agenda, the bargaining abilities of the players, etc. Thus either a_1 or a_3 (or a_0 in case of a deadlock in bargaining) can be expected (in an empirical setting) if the game is played by rational players.

We now turn our attention to the question of existence of the one-core for finite simple committee games. The one-core always exists and is nonempty for n-person finite simple committee games when $n \leqslant 4$, and we exhibit a five-person finite simple committee game for which the one-core does not exist. Before we prove the above results we need a lemma.

LEMMA 4.3. *Let Γ be a n-person simple committee game with no dummies and $n \leqslant 4$. Then there exists $i_0 \in N$ such that*

$$\{R \in \mathcal{W} : R \subset N - \{i_0\}\} = \{N - \{i_0\}\} \text{ or } \emptyset.$$

PROOF. The proof is by simple enumeration of all proper simple games with four or fewer players. Shapley [21] has listed all these games. Pick i_0 to be the "strongest" player. Note that if i_0 is a veto player then

$$\{R \in \mathcal{W} : R \subset N - \{i_0\}\} = \emptyset.$$

THEOREM 4.4. *Let Γ be a n-person finite simple game with no dummies and $n \leqslant 4$. Then $\mathcal{C}_1 \neq \emptyset$. Also there exists $x \in X$ such that $(i_0, x) \in \mathcal{C}_1$ where i_0 is as given by Lemma* 4.3.

PROOF. Define $D^1(a_0) = \{x \in X : a_0 \operatorname{dom}_R x, R \subset N - \{i_0\}\}$ where i_0 is as given by Lemma 4.3 and let $X_p = X - D^1(a_0)$. Since $a_0 \in X_p$, $X_p \neq \emptyset$. Pick any $j \in N$, $j \neq i_0$. Let $z \in X_p$ such that $x \leqslant_j z$ for each $x \in X_p$. We claim that $(i_0, z) \in \hat{\mathcal{C}}^{(i_0)}$. Note that since $z \in X_p$, z cannot be dominated by a blocking coalition not containing i_0. Suppose the claim is false, i.e., z is dominated by some $y \in X$ via a winning coalition R not containing i_0. By Lemma 4.3, $R = N - \{i_0\}$. Hence $y >_k z$ for each $k \in N - \{i_0\}$. In particular $y >_j z$, hence by our choice of z above, $y \notin X_p$ (otherwise $y \leqslant_j z$ and $y >_j z$ leads to a contradiction). Hence $y \in D^1(a_0)$, i.e., $a_0 >_k y$ for each $k \in R^1$ for some $R^1 \subset N - \{i_0\}$ such that $a_0 \in v(R^1)$. Hence $a_0 >_k y >_k z$ for each $k \in R^1$ and $a_0 \in v(R^1)$ implies that $z \in D^1(a_0)$, i.e., $z \notin X_p$, a contradiction! This proves our claim and since X is finite, $C^{(i_0)} \neq \emptyset$. Q.E.D.

The following example exhibits a five-person finite simple committee game that has no one-core.

EXAMPLE 4.2. Let $N = \{1,2,3,4,5\}$, $X = \{a_0, a_1, a_2, a_3, a_4, a_5\}$, v be given by

$$v(R) = \begin{cases} X & \text{if } R \geqslant 3, \\ \emptyset & \text{if } R \leqslant 2, \end{cases}$$

and the utility functions be given by

	u_1	u_2	u_3	u_4	u_5
a_0	0	0	0	0	0
a_1	1	2	3	4	0
a_2	2	3	4	0	1
a_3	3	4	0	1	2
a_4	4	0	1	2	3
a_5	0	1	2	3	4

A simple calculation will show that $\mathcal{C}_1$ does not exist.

Compared to the core, the stability requirement in the one-core was slightly relaxed. (However, considering the behavioral motivation behind the one-core, the stability requirement seems as strong as in the core.) As a consequence, the one-core has a stronger existence result than the core. As exhibited by the classical voters paradox (see Arrow [1]), the core can be empty when the number of players is as small as three. However, this strengthening of the existence result for the one-core is obtained in a perverse kind of way. As seen in the proof of Theorem 4.4, the existence of the one-core is shown by a player making a proposal that is the best for someone else. Like the core, the one-core is concerned only about stability and ignores whether a player really wants to make his proposal. This is a major drawback of the one-core and is illustrated by the following example.

EXAMPLE 4.3. (A discrete 3-person constant sum game.) Let $N = \{1,2,3\}$, $X = \{x = (x_1, x_2, x_3) \in E^3 : \sum_{i=1}^{3} x_i = 9,\ x_i \geqslant 0$, integer for each $i \in N\}$, v be given by

$$v(R) = \begin{cases} X & \text{if } |R| \geqslant 2, \\ \emptyset & \text{if } |R| = 1, \end{cases}$$

and the utility functions be given by

$$\text{for each } x \in X, \quad u_i(x) = x_i,\ i = 1,2,3.$$

Then, a simple calculation reveals that $\mathcal{C} = \emptyset$ and $\mathcal{C}_1 = \{(i,x) : x_i = 1,\ x \in X,\ i = 1,2,3\}$. In fact, in the continuous version of the above game (obtained by dropping the integer requirement for x_i in the definition of X), the one-core would consist of proposals wherein each player would propose giving himself a zero amount!

In the above example, the one-core gives the security level of players under the possibly pessimistic assumption that they have to move first, i.e., make the first proposal. Example 4.1 indicates a case in which moving first is advantageous, Example 4.3 a case where it's not. This kind of analysis could be very useful in studying the effect of agenda and agenda control on outcomes.[7]

An obvious question that arises is for what class of games is moving first advantageous at least as indicated by the one-core. A partial answer to this question is given by Theorems 4.1 and 4.2. Theorem 4.1 states that when the Condorcet solution exists, then the one-core must propose this point. Theorem 4.2 states that when the core is non-empty, a player cannot do any worse (and maybe do better) than the best outcome in the core by proposing his proposal as prescribed by the one-core. Thus a good thumb-rule is that for the class of games with non-empty cores, the one-core prescription, if different from the core, is certainly better than the core proposals. The kind of pathology exhibited by Example 4.3 is only possible for games with empty cores. This is because the stability requirements of the one-core (and the core) are too strong for such games. A possible way out of this impasse is to relax the stability requirement of the one-core. One approach is to use the idea of the bargaining set first developed by Aumann and Maschler [4] in the context of games with side payments. This is discussed in the next section.

5. The Bargaining Set of Committee Games

The concept of a bargaining set was first introduced by Aumann and Maschler [4] in the context of games with side payments. They defined several kinds of bargaining sets. These were generalized for games without side payments and studied by Peleg [19], Billera [5], [6], D'Aspremont [9], and Asscher [2], [3]. Since then, several other

[7] See Levine and Plott [15] for a discussion of this problem.

modifications of the bargaining set have been studied in different contexts by Shenoy [22], Rosenthal [20], Wilson [23] and Issac and Plott [13].

Here we define yet another bargaining set. This definition is relevant to the context of a committee game and is presented as an extension of the one-core. The definition presented here captures some of the ideas presented in all the references mentioned above but is quite different from all of them. Along with the definition, we present a behavioral interpretation of the bargaining set.

Suppose a player $i \in N$ introduces a motion $x \in X$ in the form of a proposal (i, x). The proposal is debated by the members. At the end of the debate there are three possible courses of action:

(1) Player i withdraws his motion. Another proposal is made and the process continues.

(2) The proposal (i, x) is uncontested and becomes the decision of the committee and the members go home.

(3) Another member $j \in N$ introduces another motion $y \in X$ and the two proposals are put to vote with the members voting for one of the two motions. The motion that wins becomes the new proposal and the process continues. In case neither motion gets a decisive vote (in case of non-strong simple committee games), the motion introduced first is considered undefeated and remains as the current proposal.[8] An *objection* against proposal (i, x) is a triple (j, S, y) such that

$$j \in S \in 2^N, y \in X, \tag{5.1}$$

$$i \notin S, \tag{5.2}$$

$$y \operatorname{dom}_S x. \tag{5.3}$$

During the process of debating the merits (or demerits) of the proposal (i, x), player j puts forward an objection to the effect: why shouldn't I introduce a motion y against the proposal (i, x)? Since $y \operatorname{dom}_S x$ (Condition 5.3)), j expects the players in coalition S to vote for y which would result in y winning against x. Condition (5.2) reflects the fact that player j cannot expect player i to cooperate with him in defeating his (player i's) own proposal. Note that Condition (5.1) and (5.2) require i and j to be two distinct players. Player j's objection is directed towards all players in $N-S$. A *counterobjection* against the objection (j, S, y) to proposal (i, x) is a triple (k, T, z) such that

$$k \in T \in 2^N, z \in X, \tag{5.4}$$

$$\text{either } k = i \text{ or } x >_k y, \tag{5.5}$$

$$z \operatorname{dom}_T y, \tag{5.6}$$

$$x >_j z, \tag{5.7}$$

$$x \leqslant_k z. \tag{5.8}$$

The counterobjection is made either by player i or by a player who stands to lose if the objection is carried out (Condition (5.5)).

The counterobjection is a reply by player k to player j to the effect: If you (player j)

[8] We have deliberately not specified all the rules. E.g. several or all the members may wish to make a proposal. We will assume that the rules of the committee will decide which proposal is considered first. These ambiguities result in the bargaining set having several proposals. Which of these are actually realized will depend on the committee rules. At this stage we wish to be as general as possible.

carry out your objection and win, then in the next round I will introduce motion z against y and since $z\ \mathrm{dom}_T\ y$ (Condition (5.6)) z will win against y. This will put you in a worse position than you were before (Condition (5.7)) and I will do no worse than what I started with (Condition (5.8)).

If a counterobjection does exist, there is a strong motivation for player j to withdraw his objection. On the other hand if there is no counterobjection, then player j has a justified objection and player i cannot expect to get his proposal accepted by the committee.

PROPOSITION 5.1. *Let (i,x) be a proposal, (j,S,y) be an objection and (i,T,z) be a counterobjection. Then $j \notin T$, $k \notin S$ and $T \cap S \neq \emptyset$.*

PROOF. $y >_j x$ and $x >_j z \Rightarrow y >_j z$. Then $z\ \mathrm{dom}_T\ y \Rightarrow j \notin T$. Conditions (5.2), (5.3) and (5.5) $\Rightarrow k \notin S$. $T \cap S = \emptyset$ contradicts condition (2.3) since $y \in v(S)$ and $z \in v(T)$.

A proposal is said to be $\mathfrak{M}$-*stable* if every objection has a counterobjection. Let $\hat{\mathfrak{M}}$ denote the set of all $\mathfrak{M}$-stable proposals. Conceivably we could have two (or more) $\mathfrak{M}$-stable proposals of the type (i,a) and (i,b). In such a case since player i is a rational player, we can trust him to introduce only those proposals in $\hat{\mathfrak{M}}$ which will maximize his utility.

A $\mathfrak{M}$-stable proposal (i,a) is said to be *maximal* if $x \leqslant_i a$ for each $x \in X$ such that $(i,x) \in \hat{\mathfrak{M}}$.

The *bargaining set* $\mathfrak{M}$ is the set of all maximal $\mathfrak{M}$-stable proposals.

EXAMPLE 5.1. Consider the 3-person committee game given in Example 4.3. The bargaining set $\mathfrak{M}$ of this game is given by

$$\mathfrak{M} = \{(1,(3,3,3)),(2,(3,3,3)),(3,(3,3,3))\}.$$

Consider the proposal $(1,(3,3,3))$. An objection by player 2 is $(2,(23),(0,4,5))$ A counterobjection by player 1 is $(1,(13),(3,0,6))$. It can easily be shown that $(1,(3,3,3))$ is a maximal $\mathfrak{M}$-stable proposal.

We shall now study the question of existence of the bargaining set. Comparing the bargaining set $\mathfrak{M}$ with the Condorcet solution α, we have:

THEOREM 5.2. *Let Γ be a committee game such that the Condorcet solution α exists. Then $\mathfrak{M} \neq \emptyset$ and is given by $\mathfrak{M} = \{(1,\alpha),(2,\alpha),\ldots,(n,\alpha)\}$.*

PROOF. It is clear that $\hat{\mathfrak{M}} \supset \{(1,\alpha),\ldots,(n,\alpha)\}$. Consider a proposal (i,x) such that $x \neq \alpha$. Since α is a condorcet solution, $\alpha\ \mathrm{dom}_R\ x$ for some $R \in 2^N$. If $i \notin R$, then (j,R,α) is an objection to (i,x) for which there is no counterobjection. Hence $(i,x) \notin \mathfrak{M}$. If $i \in R$, then if (i,x) is $\mathfrak{M}$-stable, (i,x) is not maximal since $(i,\alpha) \in \hat{\mathfrak{M}}$ and $\alpha >_i x$. Q.E.D.

Comparing the bargaining set with the core $\mathcal{C}$, we have:

THEOREM 5.3. *Let Γ be a finite committee game such that the core $\mathcal{C}$ is nonempty. Then we have*

(1) $\mathfrak{M} \neq \emptyset$,

(2) *for each $i \in N$, there exists $x \in X$ s.t. $(i,x) \in \mathfrak{M}$,*

(3) $(i,x) \in \mathfrak{M} \Leftrightarrow y \leqslant_i x$ *for each* $y \in \mathcal{C}$.

PROOF. Clearly $\{(i,x): i \in N \text{ and } x \in \mathcal{C}\} \subset \hat{\mathfrak{M}}$. Since X is finite $\mathfrak{M} \neq \emptyset \Leftrightarrow \hat{\mathfrak{M}} \neq \emptyset$. Since $\mathcal{C} \neq \emptyset$, the first two assertions hold. The third assertion follows from the fact that $\mathfrak{M}$ consists of only maximal $\mathfrak{M}$-stable proposals in $\hat{\mathfrak{M}}$.

And finally, comparing $\mathfrak{M}$ with the one-core $\mathcal{C}_1$ we have:

THEOREM 5.4. *Let Γ be a finite committee game such that the one-core $\mathcal{C}_1$ is nonempty. Then we have*
(1) $\mathfrak{M} \neq \emptyset$, *and*
(2) $(i, x) \in \mathfrak{M} \Rightarrow y \leqslant_i x$ *for each* $y \in X$ *such that* $(i, y) \in \mathcal{C}_i$.

The proof is exactly as in Theorem 5.3.

Theorems 5.2, 5.3 and 5.4 tell us what to expect from the bargaining set vis-à-vis the Condorcet solution, the core and the one-core. The bargaining set is nonempty whenever the Condorcet solution exists, or the core or the one-core is non-empty, a consequence of having relaxed the stability requirements. When the Condorcet solution exists, the bargaining set proposes this solution. When the core is nonempty, a player does no worse (and maybe better) by proposing his proposal in the bargaining set (assuming that the stability of the proposal is strong enough that he can have it accepted by the committee). The same is true with respect to the one-core. The main advantage of the bargaining set over the core and the one-core is in games with empty cores. For such games, the proposals in the bargaining set are often more reasonable than the one-core proposals as illustrated in Examples 5.1 and 5.2. When the core is nonempty, the bargaining set will often coincide with the one-core.[9]

Unlike the Aumann-Maschler bargaining set for characteristic function games, the bargaining set defined here does not take into account the possibility of coalitions among the players. Both the one-core and the bargaining set defined here are silent on the question of strategic behavior of players with regards to coalition formation. This is illustrated in the following example.

EXAMPLE 5.2. Let $N = \{1,2,3\} : X = E^2$ the two dimensional Euclidean plane; v be given by

$$v(R) = \begin{cases} X & \text{if } |R| \geqslant 2, \\ \emptyset & \text{if } |R| < 2, \end{cases}$$

and the utility function be given by

$$u_i(x) = 1/(\|b_i - x\|_2 + K)$$

where $\| \; \|_2$ denotes the Euclidean distance norm, $b_i \in E^2$ is player i's "bliss point" and K is a positive constant. In other words, each player has circular indifference contours with the maximum utility at his specified bliss point and with utility decreasing as the distance from the bliss point increases. Let the configuration of the bliss points be as shown in Figure 5.1 below.

Representing the players generically by i, j and k such that $i \neq j \neq k$, note that the convex hull of $\{b_i, b_j\}$ dominates all other points in E^2 via $\{i, j\}$. Hence the core of the game is empty. Using (4.1), $\hat{C}^{(i)}$ = convex hull of $\{b_j, b_k\}$. Then by (4.2), $C^{(i)}$ is that subset of $\hat{C}^{(i)}$ that lies closest to b_i. Thus the one-core is given by

$$\mathcal{C}_1 = \{(1, b_3), (2, b_3), (3, x_4)\}.$$

As mentioned before, as the core is empty, the one-core indicates the security level of

[9] In [16], Maschler gives an example of a game with a nonempty core whose bargaining set is bigger than the core. For this game, Maschler argues that the bargaining set is more intuitive than the outcomes contained in the core.

the players. The bargaining set of this game is given by

$$\mathfrak{M} = \{(1, x_0), (2, x_0), (3, x_0)\}.$$

Where x_0 as shown in Figure 5.1 is characterized as follows. x_0 is the unique point in the interior of the triangle b_1, b_2, b_3 such that it lies on the intersection of circles with centers at b_i and b_j and furthermore these circles are tangent to a common circle with centre at b_k, simultaneously for $i = 1, 2, 3$ and $i \neq j \neq k$. The point x_0 can be considered as the "center" of the circular triangle x_1, x_2 and x_3 where these are a unique set of points such that there is one indifference contour through two of these points for each player and the two circles passing through any one of the points are tangent.[10]

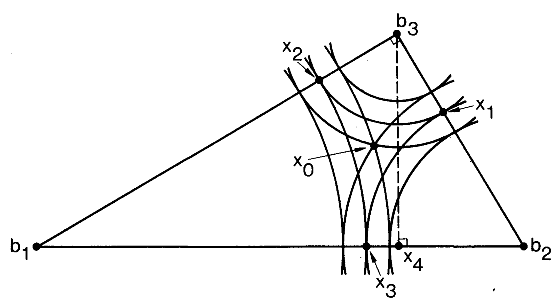

FIGURE 5.1. The One-Core and the Bargaining Set of the Game in Example 5.2.

The bargaining set is not always nonempty. For the committee game presented in Example 4.3, $\mathfrak{M} = \emptyset$. Where do we go from here? One possibility is to further relax the stability requirements of the bargaining set by modifying the definition of a counterobjection. We could drop Condition (5.7) or Condition (5.8) or both. We could also alter the definition of an objection by requiring an additional condition as follows

$$x >_i y. \tag{5.9}$$

This method is not entirely satisfactory. Each solution concept implies a definition of "rational behavior." The "rational behavior" implied by these modified bargaining sets is difficult to accept in real life.

6. Conclusion

Four different solution concepts were studied in relation to committee games. These are the Condorcet solution, the core, the one-core and the bargaining set. (The

[10]To compare our solution with some other solution concepts, note that the set of points $\{x_1, x_2, x_3\}$ constitutes the main simple von Neuman-Morgenstern solution [18] and also the competitive solution due to McKelvey, Ordeshook and Winer [17]. In general, these solution concepts indicate a set of points that is collectively stable as a set, i.e., individual points in this set are not stable by themselves. How to use these solutions as a prescription for individual rational behavior is not very clear. If, e.g., player 2 attempts to propose x_1, after enlisting player 3's cooperation as a coalition partner, player 1 can possibly undermine this partnership by proposing b_3. Note that each of the points in the set $\{x_1, x_2, x_3\}$ violates some player's security level as indicated by the one-core. On the other hand the proposals in our bargaining set are individually stable and don't make use of coalitions (implied or explicit) to justify stability. See Fiorina and Plott [11], Issac and Plott [12], and Laing and Olmsted [14] for an empirical study of committee decisions under majority rule.

bargaining set referred to here is defined in §5 specifically for committee games and is different from the various bargaining sets that have been defined in the literature.) There is a tradeoff between stability and existence. The Condorcet solution has the strongest stability requirement but the weakest existence result. Whenever the Condorcet solution exists, all these solution concepts indicate the same solution. For games with nonempty cores, the one-core proposals are better than the core (if the two indicate different solutions) and share the stability properties of the core. For games with empty cores, the one-core indicates the security level of the players. For such games, the bargaining set with its relaxed stability requirements indicates more reasonable proposals than the one-core. The bargaining set at the other extreme has the weakest stability requirements among four solutions but the strongest existence result. However, the existence result is not strong enough to guarantee nonemptiness for all games. In the Condorcet solution and the core, the players look ahead only one "step" (only objections are considered) while the bargaining set the players anticipate two "steps" (an objection represents one step, and a counterobjection represents the second step) at a time. The one-core represents some intermediate position in this respect. Perhaps one approach is to define a solution concept that looks far ahead enough to guarantee existence and non-emptiness for all games.[11]

[11]This research was supported in part by the United States Army under contract No. DAAG29-75-C-0024 and by the National Science Foundation under Grant No. MCS75-17385 A01 at the University of Wisconsin and also by University of Kansas General Research allocation No. 3319-X0-0038 and the School of Business Research Fund provided by the Fourth National Bank and Trust Company, Wichita.

The author is grateful to Professor William F. Lucas, Philip D. Straffin and a number of anonymous referees for valuable comments and suggestions.

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